Synchronization and Multistability in Oscillator Networks and Power Grids

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Pendulum clocks: “an odd kind of sympathy”  
[Christiaan Huygens, Horologium Oscillatorium, 1673]

Models for coupled oscillators:  
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Models for coupled oscillators:  

Kuramoto model

1. n-oscillators with phases $\theta_i$,
2. with natural frequencies $\omega_i \in \mathbb{R}$,
3. coupling with strength $a_{ij} = a_{ji}$.

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j).$$
Application: Active power dynamics

- generators and inverters and loads
Application: Active power dynamics

1. **Generators** and **inverters and loads**

2. **Physics:**
   1. Kirchhoff and Ohm laws
   2. **Quasi-sync:** voltage and phase $V_i$, $\theta_i$
      active power $p_i$
Application: Active power dynamics

1 generators and inverters and loads

2 physics:
   1 Kirchhoff and Ohm laws
   2 quasi-sync: voltage and phase $V_i$, $\theta_i$
     active power $p_i$

3 simplifying assumptions:
   1 lossless and inductive lines with admittances $Y_{ij}$
   2 decoupling of phase and voltage dynamics
Application: Active power dynamics

Structure-Preserving Model [A. Bergen & D. Hill, 1981]:

Active power dynamics

| Generators: | $M_i \ddot{\theta}_i + D_i \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$ |
| Inverters:  | $\Lambda_i \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$ |
| Loads:      | $\tau_i \dot{\theta}_i = p_i - \sum_j a_{ij} \sin(\theta_i - \theta_j)$ |

where

Active power capacity of line $(i, j)$: $a_{ij} = |Y_{ij}| V_i V_j$
Structure-Preserving Model [A. Bergen & D. Hill, 1981]:

**Active power dynamics**

<table>
<thead>
<tr>
<th>Category</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
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where

**Active power capacity of line** $(i, j)$: $$a_{ij} = |Y_{ij}| V_i V_j$$

**Frequency synchronization** = **Operating point** = **Equilibriums**

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$
Synchronization problem

Frequency synchronization = Operating point = Equilibriums

\[ p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

Key questions

Given the network and the power profile:

Q1: does there exist a stable equilibrium point?

Q2: is this stable equilibrium point unique?

Q3: how to measure the robustness of the synchronization?
Q1: Existence of a sync state:

\[ \dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

\[ a_{ij} = 1 \]

\[ P = 1 \]

\[ P = 2.5 \]
Q1: Existence of a sync state:

\[ \dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

- sync threshold: “power transmission” vs. “coupling”
- quantify: “power transmission” < “coupling”
- as a function of network parameters
Weighted undirected graph with $n$ nodes and $m$ edges:

**Incidence matrix:** $n \times m$ matrix $B$ s.t. $(B^\top p)_{(ij)} = p_i - p_j$

**Edge weight matrix:** $m \times m$ diagonal matrix $A$

**Laplacian matrix:** $L = BAB^\top$

**Equilibrium point:** $p = BA \sin(B^\top \theta)$

**Algebraic connectivity:** $\lambda_2(L) = $ second smallest eig of $L$

**Cycle space:** $\text{Ker}(B) = $ span of all the cycle vectors
Given a network and $p$, does there exist angles?

$$p = BA \sin(B^\top \theta)$$

synchronization arises if

**power transmission < coupling strength**
Given a network and $p$, does there exist angles?

$$p = B A \sin(B^\top \theta)$$

Synchronization arises if

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Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$$\|B^\top p\|_2 < \lambda_2(L) \quad \text{for all graphs}$$

(Old 2-norm $T$)

(Old $\infty$-norm $T$)
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$$\|B^\top p\|_2 < \lambda_2(L) \quad \text{for all graphs}$$

$$\|B^\top L^\dagger p\|_\infty < 1 \quad \text{for trees, complete}$$

(Old $2$-norm T)

(Old $\infty$-norm T)
\[ \mathbb{R}^m \quad = \quad \text{Im}(B^\top) \quad \oplus \quad \text{Ker}(BA) \]

- \( \text{edge space} \)
- \( \text{cutset space} \)
- \( \text{weighted cycle space} \)
$\mathbb{R}^m$  
\underline{edge space}  

$= \ \underline{\text{Im}(B^\top)} \ \oplus \ \underline{\text{Ker}(BA)}$  
\underline{cutset space} \underline{weighted cycle space}  

$\mathcal{P} = B^\top L^\dagger BA$  
\underline{cutset projection}  

= oblique projection onto $\text{Im}(B^\top)$  
parallel to $\text{Ker}(BA)$
\[
\mathbb{R}^m = \text{Im}(B^\top) \oplus \text{Ker}(BA) \\
\mathcal{P} = B^\top L^\dagger BA = \text{oblique projection onto Im}(B^\top) \parallel \text{Ker}(BA)
\]

1. If \( G \) acyclic, then \( \mathcal{P} = I_m \)
2. If \( G \) unweighted, then \( \mathcal{P} \) is an orthogonal projection
3. If \( R_{\text{eff}} \in \mathbb{R}^{n \times n} \) are effective resistances, then \( \mathcal{P} = -\frac{1}{2} B^\top R_{\text{eff}} BA \)
Find sufficient conditions on $B, A, p$ s.t. there exists a solution $\theta$ to:

$$p = BA \sin(B^\top \theta)$$
Rewriting the equilibrium equation

Find sufficient conditions on $B, A, p$ s.t. there exists a solution $\theta$ to:

$$p = B A \sin(B^\top \theta)$$

Key idea: Node vs. Edge

$$p = B A \sin(B^\top \theta) \quad \updownarrow$$

$$B^\top L^\dagger p = \mathcal{P} \sin(B^\top \theta)$$

Node balance eq. $\mathbb{R}^n$

Edge balance eq. $\mathbb{R}^m$
Rewriting the equilibrium equation

Find sufficient conditions on $B, A, p$ s.t. there exists a solution $\theta$ to:

$$p = BA \sin(B^\top \theta)$$

Key idea: Node vs. Edge

\[ p = BA \sin(B^\top \theta) \quad \Downarrow \quad B^\top L^\dagger p = P \sin(B^\top \theta) \]

- **Edge variables**: $x = B^\top \theta$ and $z = B^\top L^\dagger p$

Find sufficient conditions on $z \in \text{Im}(B^\top)$ s.t. there exists solution $x$ to:

$$z = P \sin(x) = P[\text{sinc}(x)]x$$
2 look for $x \in B_p(\gamma) = \{ x \mid \| x \|_p \leq \gamma \}$ solving

$$P[\text{sinc}(x)]x = z \iff x = (P[\text{sinc}(x)])^{-1}z =: h(x)$$
look for $x \in B_p(\gamma) = \{x \mid \|x\|_p \leq \gamma\}$ solving

$$\mathcal{P}[\text{sinc}(x)]x = z \iff x = (\mathcal{P}[\text{sinc}(x)])^{-1}z =: h(x)$$

define min amplification factor of $\mathcal{P}[\text{sinc}(x)] : \text{Im}(B^\top) \rightarrow \text{Im}(B^\top)$

$$\alpha_p(\gamma) := \min_{\|x\|_p \leq \gamma} \min_{\|y\|_p = 1} \|\mathcal{P}[\text{sinc}(x)]y\|_p$$
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\[
\alpha_p(\gamma) := \min_{\|x\|_p \leq \gamma} \min_{\|y\|_p = 1} \|P[sinc(x)]y\|_p
\]
\[
\|z\|_p \leq \gamma \alpha_p(\gamma) \implies h \text{ satisfies Brouwer on } B_p(\gamma)
\]
Equilibrium angles (neighbors within $\gamma$ arc) exist if, in some $p$-norm,

$$\left\| B^\top L^\dagger p \right\|_p \leq \gamma \alpha_p(\gamma)$$

for all graphs (New $p$-norm $T$)

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- **Caveat:** $\alpha_p(\gamma)$ requires solving a non-convex optimization problem!
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- Algebraic manipulation $\implies$ lower bound $\alpha_p(\gamma)$.

For $p = \infty$, the new test for all graphs

$$\|B^\top L^\dagger p\|_\infty \leq g(\|\mathcal{P}\|_\infty) \quad \text{(New } \infty\text{-norm T)}$$
Function $g$ is strictly decreasing

$g : [1, \infty) \rightarrow [0, 1]$

$g(x) = \frac{y(x) + \sin(y(x))}{2} - x\frac{y(x) - \sin(y(x))}{2}$

$y(x) = \arccos\left(\frac{x-1}{x+1}\right)$
$K_C = \text{critical coupling of Kuramoto model, computed via MATLAB } \texttt{fsolve}$

$K_T = \text{smallest value of scaling factor for which test } T \text{ fails}$

<table>
<thead>
<tr>
<th>Test Case</th>
<th>Critical ratio $K_T/K_C$</th>
<th>$g(|P|_{\infty})$</th>
<th>Approx.test</th>
<th>$\alpha_{\infty}(\pi/2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Old 2-norm $T$</td>
<td>New $\infty$-norm $T$</td>
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</tr>
<tr>
<td>IEEE 9</td>
<td>16.54 %</td>
<td>73.74 %</td>
<td>92.13 %</td>
<td>85.06 %†</td>
</tr>
<tr>
<td>IEEE RTS 24</td>
<td>3.86 %</td>
<td>53.44 %</td>
<td>89.48 %</td>
<td>89.48 %†</td>
</tr>
<tr>
<td>New England 39-bus</td>
<td>2.97 %</td>
<td>67.57 %</td>
<td>100 %</td>
<td>100 %†</td>
</tr>
<tr>
<td>IEEE 118</td>
<td>0.29 %</td>
<td>43.70 %</td>
<td>85.95 %</td>
<td>—*</td>
</tr>
<tr>
<td>IEEE 300</td>
<td>0.20 %</td>
<td>40.33 %</td>
<td>99.80 %</td>
<td>—*</td>
</tr>
<tr>
<td>Polish 2383</td>
<td>0.11 %</td>
<td>29.08 %</td>
<td>82.85 %</td>
<td>—*</td>
</tr>
</tbody>
</table>

† $\texttt{fmincon}$ has been run for 100 randomized initial phase angles.
* $\texttt{fmincon}$ does not converge.
Multistable equilibrium points

**Q2:** Is the equilibrium point unique?

\[ \dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

\[ \theta_0 = [-1, 1, -2, 0, 1.5, 0.5]^T \]

\[ \theta_0 = [0, 1.2, 2.2, 3.9, 4.8, 1]^T \]
Mutlistable equilibrium points

**Q2**: Is the equilibrium point unique?

\[ \dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

\[ a_{ij} = 1 \]

\[ P = 1/4 \]

- multistable sync: “cycle structure” and “state space”
- quantify: “cycle structure” vs “multistable sync”
Key question

How to localize stable equilibrium points?

Winding number
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How to localize stable equilibrium points?

Winding number

Nodal variables in $\mathbb{R}^3$

$$\sum_{i=1}^{3} (x_i - x_{i+1}) = 0.$$ 

Nodal variables in $\mathbb{T}^3$

$$\sum_{i=1}^{3} (\theta_i - \theta_{i+1}) = 2\pi w_\sigma(\theta),$$ 

where $w_\sigma(\theta) \in \mathbb{Z}$, the winding number.
Winding partition of the $n$-torus

Winding vector

Given a graph $G$ with a cycle basis $\Sigma = \{\sigma_1, \ldots, \sigma_{m-n+1}\}$ and $\theta \in \mathbb{T}^n$.

Winding vector: $w_\Sigma(\theta) = [w_{\sigma_1}(\theta), \ldots, w_{\sigma_{m-n+1}}(\theta)]^\top \in \mathbb{Z}^{m-n+1}$
Winding partition of the $n$-torus

### Winding vector

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### Winding cells: equivalence classes

Given a graph $G$ with a cycle basis $\Sigma$. For every $\mathbf{u} \in \mathbb{Z}^{m-n+1}$

(Winding cell $\mathbf{u}$) = all $\theta \in \mathbb{T}^n$ s.t. $w_\Sigma(\theta) = \mathbf{u}$.
At-most uniqueness in winding cells

Winding partition of $n$-torus

\[ T^n = \bigcup_{u \in \mathbb{Z}^{m-n+1}} \text{(Winding cell } u) \]

\[ p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

There is either zero or one stable equilibrium point (neighbors within $\pi/2$ arc) in each winding cell.
At-most uniqueness in winding cells

Winding partition of $n$-torus

$T^n = \bigcup_{u \in \mathbb{Z}^{m-n+1}} (\text{Winding cell } u)$

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There is either zero or one stable equilibrium point (neighbors within $\pi/2$ arc) in each winding cell.
Key question

How to check if we have a stable equilibrium point inside a winding cell?
**Key question**

How to check if we have a stable equilibrium point inside a winding cell?

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<th><strong>Nodal Balance</strong></th>
<th><strong>Edge Balance</strong></th>
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</table>
| \[ p = BA \sin(B^\top \theta), \] neighbors within \( \gamma \) arc | \[ \begin{align*}
B^\top L^\dagger p &= P_{\text{cut}} \eta, \\
\| \eta \|_\infty &\leq \sin(\gamma), \\
P_{\text{cyc}}(\arcsin(\eta) - 2\pi C_\Sigma^\dagger u) &= 0_m.
\end{align*} \] |

- \( P_{\text{cut}} \) is the projection onto the cutset space
- \( P_{\text{cyc}} \) is the projection onto the cycle space
- Separates the cycle flows and cutset flows
- Highlights the role of the winding vector
Existence/Computation

Iterations for edge balance equations

\[ \eta^{(k+1)} = B^\top L^\dagger p + \mathcal{P}_{\text{cyc}} \left( \eta^{(k)} - \cos(\gamma)(\arcsin(\eta^{(k)}) - 2\pi C_{\Sigma}^\dagger u) \right). \]

- Start from any \( \eta^{(0)} \in \mathbb{R}^m \)
- The sequence is **contractive** and always converges (to a vector \( \eta^* \))
- **If** \( \|\eta^*\|_\infty > \sin(\gamma) \): no stable equilibrium point in winding cell \( u \).
- **If** \( \|\eta^*\|_\infty \leq \sin(\gamma) \): one stable equilibrium point in winding cell \( u \);

\[ \theta^* = L^\dagger B A(\arcsin(\eta^*) - 2\pi C_{\Sigma}^\dagger u) \]
Summary and future work

Contributions

- geometry of cutset projection operator
- family of sufficient sync conditions
- partition of $n$-torus based on winding vector
- localize the equilibrium points using winding cells

Future research

- close the gap between sufficient and necessary conditions
- region of attraction of stable equilibrium points
- generalizations to other oscillator models.
Summary and future work

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