Local reachability of control systems in étalé Lie groupoids

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Abstract

The notion of a pseudogroup provides an apt generalisation of the set of flows associated with a control system. The notion of a groupoid provides an apt generalisation of a pseudogroup. With this as motivation, control systems in Lie groupoids are considered, and a list of conditions is provided that are equivalent to local reachability for such control systems.

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1. Introduction

In this paper we shall study a generalisation of the problem of local controllability (which we call "reachability" in this paper) that is important in control theory. The generalisation is developed in two directions. First, rather than considering flows of vector fields as is typically the situation in control theory, we consider subsets of the pseudogroup of local diffeomorphisms. This generalisation is not an uncommon one, as it appears in the fundamental work of Sussmann [1973] on the Orbit Theorem and, for example, in the study of reachability of Kupka and Sallet [1983]. The other generalisation we make is from pseudogroups to Lie groupoids. This generalisation is considered in the work of Stefan [1974a, 1974b] on the Orbit Theorem, but has not really been developed by the control community. In this paper we consider local reachability in the setting of Lie groupoids.

Of course, much work has been done on the problem of reachability in the geometric control literature. Our approach is quite different from existing approaches. One way to characterise the existing literature is that it consists of various sufficient conditions and necessary conditions, typically Lie algebraic in nature, with attempts to close the gaps between these. As is discussed by Lewis [2014, Chapter 1], the framework normally used to study control systems is not feedback-invariant, and as a result the theorems obtained are not feedback-invariant. That is, it can be the case that a theorem is conclusive or inconclusive about the same system, depending on how controls are parameterised. That is to say, many results on reachability are results on a control system *with a fixed control paramaterisation*. This is most easily understood by considering the method of nilpotent approximation developed initially by Sussmann [1983, 1987], and used by many others subsequently [e.g., Bianchini and Kawski 2003, Bianchini and Stefani 1993, Hermes 1991, Kawski 1998]. In

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developing reachability conditions based on such a methodology, the conclusions will depend in unpredictable ways on how one represents a system [Lewis 2014, Examples 1.1 and 1.2]. By contrast, here we develop inherently feedback-invariant necessary and sufficient conditions for reachability by looking at germs and jets of local diffeomorphisms of the system as mappings between germs and jets of functions. The conditions we obtain come in two flavours. First we obtain algebraic conditions that can be interpreted as point separating conditions for families of linear maps defined by our system's local diffeomorphisms. Second, we obtain a separating hyperplane condition that is reminiscent to the method frequently used to show that a system is not locally reachable, e.g., [Sussmann 1983, Proposition 6.3], [Kawski 1990, Example 5.2].

In all cases, the conditions have a rather different character than the Lie bracket conditions normally encountered in the reachability literature. The intent is to develop a diffeomorphism framework that can be subsequently used to develop conditions on vector fields defining the flows for the system. In the last section of the paper we make explicit the connection between our notions of reachability and the notions most commonly used in control theory. However, the approach itself is interesting, independent of its connection to more standard approaches, so we feel that it is worth independent explication.

1.1. An outline of the paper. As the paper has to do with reachability of systems defined on Lie groupoids, and since this is not the usual setup for control theory, in Section 2 we illustrate exactly how our formalism generalises the usual formalism. We do this by first illustrating the well-known idea that pseudogroups arise naturally from flows of vector fields, even vector fields with measurable time-dependence, thanks to the results of Jafarpour and Lewis [2014]. Then we illustrate that pseudogroups admit the generalisation to étalé Lie groupoids. The Lie groupoid formalism is useful because it allows for our particularly simple notion of control system in Definition 3.1. In Section 3.2 we see that this notion of system allows for many of the standard notions for reachability to make sense, e.g., accessibility and reachability. In Section 3.3 we provide four conditions that are equivalent to local reachability for certain large classes of systems. In Section 3.4 we illustrate explicitly how our reachability definitions subsume any of the standard notions of reachability one can find in the control theory literature.

1.2. Notation and background.

1.2.1. The basics. We use standard set theoretic conventions, with the exception that when we write $A \subset B$ we mean strict inclusion of sets. When we wish to allow for the possibility that $A = B$ we write $A \subseteq B$.

By $\mathbb{Z}$ we denote the set of integers, with $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denoting the sets of positive and nonnegative integers, respectively. By $\mathbb{R}$ we denote the set of real numbers, with $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denoting the positive real numbers and nonnegative real numbers, respectively.

1.2.2. Geometric notation. We mostly use the geometric notation and conventions of [Abraham, Marsden, and Ratiu 1988]. Manifolds will be smooth (i.e., of class C^∞) or real analytic (i.e., of class C^ω), and will be assumed to be Hausdorff and second countable, unless stated otherwise. (We will drop these conditions for the total space of a Lie groupoid.) The tangent bundle of a manifold M is denoted by $\pi_{\mathsf{TM}}\colon \mathsf{TM} \to \mathsf{M}$ and the cotangent bundle by $\pi_{\mathsf{T}^*\mathsf{M}}\colon \mathsf{T}^*\mathsf{M} \to \mathsf{M}$. For $\nu \in \{\infty, \omega\}$, by $\Gamma^\nu(\mathsf{TM})$ we denote the set of C^ν-vector fields on M which we regard as a vector space over $\mathbb{R}$. By $\mathrm{C}^\nu(\mathsf{M})$ we denote the set of $\mathbb{R}$-valued functions on M of class C^ν. By $\mathrm{C}^\nu(\mathsf{M};\mathsf{N})$ we denote the set of mappings of class C^ν between manifolds M and N. If $\Phi \in \mathrm{C}^\nu(\mathsf{M};\mathsf{N})$ and if $g \in \mathrm{C}^\nu(\mathsf{N})$, then $\Phi^* f = f \circ \Phi \in \mathrm{C}^\nu(\mathsf{M})$ is the pull-back of f by Φ.

For a vector bundle E, $\mathrm{T}^k(\mathsf{E})$ denotes the k-fold tensor product of E and $\mathrm{S}^k(\mathsf{E})$ denotes the degree k symmetric tensor algebra.

We let $\mathscr{C}^\infty_{\mathsf{M}}$ denote the sheaf of smooth functions on a smooth manifold M. The stalk at $x \in \mathsf{M}$ we denote by $\mathscr{C}^\infty_{x,\mathsf{M}}$, and a typical element of the stalk we denote by $[f]_x$. We have the map

$$\begin{aligned} \mathrm{ev}_x\colon \mathscr{C}^\infty_{x,\mathsf{M}} &\to \mathbb{R} \\ [f]_x &\mapsto f(x). \end{aligned}$$

We shall also use the symbol ev_x to denote the mapping

$$\begin{aligned} \mathrm{ev}_x\colon \mathrm{C}^\infty(\mathsf{M}) &\to \mathbb{R} \\ f &\mapsto f(x), \end{aligned}$$

the exact meaning being clear from context. We will at various times use the fact that a smooth manifold M is embedded in $\mathrm{C}^\infty(\mathsf{M})'$, the dual of $\mathrm{C}^\infty(\mathsf{M})$ with the weak-$*$ topology [e.g., Agrachev and Sachkov 2004, Proposition 2.1]. Explicitly, this embedding is defined by the mapping $x \mapsto \mathrm{ev}_x$, where $\mathrm{ev}_x \in \mathrm{C}^\infty(\mathsf{M})'$ is defined by $\mathrm{ev}_x(f) = f(x)$.

1.2.3. Jet bundles. We will work with jet bundles. For manifolds M and N, and $k \in \mathbb{Z}_{\geq 0}$, by $\mathsf{J}^k(\mathsf{M};\mathsf{N})$ we denote the space of k-jets of mappings from M to N. By $\rho^k_l\colon \mathsf{J}^k(\mathsf{M};\mathsf{N}) \to \mathsf{J}^l(\mathsf{M};\mathsf{N})$, $k \geq l$, we denote the projection. For $(x,y) \in \mathsf{M}\times\mathsf{N}$, $\mathsf{J}^k_{(x,y)}(\mathsf{M};\mathsf{N})$ denotes the set of k-jets of mappings that map x to y. We denote by $\mathsf{J}^k_x(\mathsf{M};\mathbb{R})$ the k-jets of functions which map $x \in \mathsf{M}$ to any value in $\mathbb{R}$. For a mapping $\Phi \in \mathrm{C}^\infty(\mathsf{M};\mathsf{N})$, we denote by $j_k\Phi\colon \mathsf{M} \to \mathsf{J}^k(\mathsf{M};\mathsf{N})$ the k-jet of Φ.

Infinite jets are defined as inverse limits. Thus

$$\mathsf{J}^\infty_{(x,y)}(\mathsf{M};\mathsf{N}) = \{\phi\colon \mathbb{Z}_{\geq 0} \to \cup_{k\in\mathbb{Z}_{\geq 0}} \mathsf{J}^k_{(x,y)}(\mathsf{M};\mathsf{N}) \mid \rho^{k+1}_k(\phi(k+1)) = \phi(k),\ k \in \mathbb{Z}_{\geq 0}\}.$$

We take

$$\mathsf{J}^\infty(\mathsf{M};\mathsf{N}) = \mathop{\mathring{\bigcup}}_{(x,y)\in\mathsf{M}\times\mathsf{N}} \mathsf{J}^\infty_{(x,y)}(\mathsf{M};\mathsf{N}),$$

and define $\rho^\infty_k\colon \mathsf{J}^\infty(\mathsf{M};\mathsf{N}) \to \mathsf{J}^k(\mathsf{M};\mathsf{N})$ by $\rho^\infty_k(\phi) = \phi(k)$. We then regard $\mathsf{J}^\infty(\mathsf{M};\mathsf{N})$ as a topological space with the inverse limit topology, i.e., the weakest topology for which the mappings ρ^∞_k, $k \in \mathbb{Z}_{\geq 0}$, are continuous.

1.2.4. Topologies for spaces of vector fields. While this is not a paper about local reachability of control systems defined by vector fields, we will make connections to these sorts of systems in order to anchor our results to familiar things. In doing so, we will make use of recent work by Jafarpour and Lewis [2014] on locally convex topologies for spaces of smooth and real analytic vector fields. (Jafarpour and Lewis also describe topologies for spaces of finitely differentiable, locally Lipschitz, and holomorphic vector fields, but we will not make use of these here.) To describe these topologies, we will merely provide the seminorms that characterise them.

To define these seminorms we introduce appropriate fibre norms for jet bundles of the tangent bundle. Let $\nu \in \{\infty, \omega\}$. We suppose that the C^ν-manifold M has a C^ν-Riemannian metric $\mathbb{G}$ and a C^ν-affine connection ∇. The existence of these for $r = \infty$ is classical and for $r = \omega$ is proved in [Jafarpour and Lewis 2014, Lemma 2.3] using the embedding theorem of Grauert [1958] for real analytic manifolds. Let $\mathrm{T}^k(\mathsf{T}^*\mathsf{M})$ denote the k-fold tensor product of $\mathsf{T}^*\mathsf{M}$ and let $\mathrm{S}^k(\mathsf{T}^*\mathsf{M})$ denote the symmetric tensor bundle. First note that ∇ defines a connection in $\mathsf{T}^*\mathsf{M}$ by duality. Then ∇ defines a connection ∇^k on $\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM}$ by asking that the Leibniz Rule be satisfied for the tensor product. Then, for a smooth vector field X, we denote

$$\nabla^{(k)} X = \nabla^k \cdots \nabla^1 \nabla X,$$

which is a smooth section of $\mathrm{T}^{k+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM}$. By convention we take $\nabla^0 X = \nabla X$ and $\nabla^{(-1)} X = X$. (The funny numbering makes this agree with the constructions in [Jafarpour and Lewis 2014, §2.1].)

We then have a map

$$\begin{aligned} S^k_\nabla \colon \mathsf{J}^k\mathsf{TM} &\to \oplus_{j=0}^k \mathrm{S}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM} \\ j_k X(x) &\mapsto (X(x), \mathrm{Sym}_1 \otimes \mathrm{id}_{\mathsf{TM}}(\nabla X)(x), \dots, \mathrm{Sym}_k \otimes \mathrm{id}_{\mathsf{TM}}(\nabla^{(k-1)} X)(x)), \end{aligned} \tag{1.1}$$

which can be verified to be an isomorphism of vector bundles [Jafarpour and Lewis 2014, Lemma 2.1]. Here $\mathrm{Sym}_k \colon \mathrm{T}^k(\mathsf{V}) \to \mathrm{S}^k(\mathsf{V})$ is defined by

$$\mathrm{Sym}_k(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Now we note that inner products on the components of a tensor product induce in a natural way an inner product on the tensor product [Jafarpour and Lewis 2014, Lemma 2.3]. Thus, if we suppose that we have a Riemannian metric $\mathbb{G}$ on M, there is induced a natural fibre metric $\mathbb{G}_k$ on $\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{TM}$ for each $k \in \mathbb{Z}_{\geq 0}$. We then define a fibre metric $\overline{\mathbb{G}}_k$ on $\mathsf{J}^k\mathsf{TM}$ by

$$\overline{\mathbb{G}}_k(j_k X(x), j_k Y(x)) \\ = \sum_{j=0}^k \mathbb{G}_j\Big(\frac{1}{j!}\mathrm{Sym}_j \otimes \mathrm{id}_{\mathsf{TM}}(\nabla^{(j-1)} X)(x), \frac{1}{j!}\mathrm{Sym}_j \otimes \mathrm{id}_{\mathsf{TM}}(\nabla^{(j-1)} Y)(x)\Big).$$

(The factorials are required to make things work out with the real analytic topology.) The corresponding fibre norm we denote by $\|\cdot\|_{\overline{\mathbb{G}}_k}$.

Now we are in a position to define the seminorms that characterise the topologies on $\Gamma^\infty(\mathsf{TM})$ and $\Gamma^\omega(\mathsf{TM})$. For the smooth case, for a compact set $K \subseteq \mathsf{M}$ and $k \in \mathbb{Z}_{\geq 0}$, we denote

$$p^\infty_{K,k}(X) = \sup\{\|X(x)\|_{\overline{\mathbb{G}}_k} \mid x \in K\}, \qquad X \in \Gamma^\infty(\mathsf{TM}).$$

The set of seminorms $p^\infty_{K,k}$, $K \subseteq \mathsf{M}$ compact, $k \in \mathbb{Z}_{\geq 0}$, defines a locally convex topology for $\Gamma^\infty(\mathsf{TM})$. To define the seminorms for $\Gamma^\omega(\mathsf{TM})$, we denote by $\mathrm{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ the set of sequences $(a_j)_{j\in\mathbb{Z}_{\geq 0}}$ in $\mathbb{R}_{>0}$ converging to 0. Then, for a compact set $K \subseteq \mathsf{M}$ and for $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, we denote

$$p^{\omega}_{K,\boldsymbol{a}}(X) = \sup\{a_0 a_1 \cdots a_k \|X(x)\|_{\overline{\mathbb{G}}_k} \mid x \in K,\ k \in \mathbb{Z}_{\geq 0}\}, \qquad X \in \Gamma^{\omega}(\mathsf{TM}).$$

The seminorms $p^{\omega}_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, define a locally convex topology for $\Gamma^{\omega}(\mathsf{TM})$. In order to allow simultaneous treatment of the smooth and real analytic cases, for a compact subset $K \subseteq \mathsf{M}$, we will denote by p_K one of the seminorms $p^{\infty}_{K,k}$, $k \in \mathbb{Z}_{\geq 0}$, or $p^{\omega}_{K,\boldsymbol{a}}$, $\boldsymbol{a} \in \mathrm{c}_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, accepting a slight loss of precision.

1.2.5. Time-varying vector fields. The preceding locally convex topologies have many valuable attributes, and we refer to [Jafarpour and Lewis 2014] for a detailed discussion of these. For our purposes, these topologies permit a characterisation of useful classes of time-varying vector fields. This allows one to consider control systems with locally essentially bounded or locally integrable controls, rather than merely piecewise constant controls.

The definitions rely on notions of measurability and integrability for mappings into $\Gamma^{\nu}(\mathsf{TM})$, $\nu \in \{\infty, \omega\}$. We shall give very terse definitions for these, referring to [Jafarpour and Lewis 2014] for details. Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A mapping $X\colon \mathbb{T} \to \Gamma^{\nu}(\mathsf{TM})$ is ***measurable*** if $X^{-1}(\mathcal{O})$ is Lebesgue measurable for every open set $\mathcal{O} \subseteq \Gamma^{\nu}(\mathsf{TM})$. A measurable mapping $X\colon \mathbb{T} \to \Gamma^{\nu}(\mathsf{TM})$ is ***locally Bochner integrable*** if $p_K \circ X$ is locally Lebesgue integrable for every seminorm p_K, $K \subseteq \mathsf{M}$ compact. By $\mathrm{LI}\Gamma^{\nu}(\mathbb{T};\mathsf{TM})$ we denote the set of locally Bochner integrable mappings from $\mathbb{T}$ to $\Gamma^{\nu}(\mathsf{TM})$.

Note that, if $X \in \mathrm{LI}\Gamma^{\nu}(\mathbb{T};\mathsf{TM})$, then we have the time-varying differential equation $\xi'(t) = X(t)(\xi(t))$. Jafarpour and Lewis [2014, Theorems 6.6 and 6.26] show that the usual existence and uniqueness results hold for these differential equations. But, far more significantly, they show that the dependence of flows on initial condition is C^{ν}. By $t \mapsto \Phi^X_{t,t_0}(x_0)$ we denote the integral curve for X passing through x_0 at time t_0. We denote

$$D_X = \{(t, t_0, x_0) \in \mathbb{R} \times \mathbb{R} \times \mathsf{M} \mid \Phi^X_{t,t_0}(x_0) \text{ exists}\}.$$

The fact that flows for vector fields from $\mathrm{LI}\Gamma^{\nu}(\mathbb{T};\mathsf{TM})$ depend on initial condition in a C^{ν} manner will allow us to connect our groupoid theory to standard control theory at a level of generality that has hitherto not been possible. Normally, when applying pseudogroup theory (and so the natural extension of this to Lie groupoids), one has to restrict to what amounts to piecewise constant controls, since in this case one is ensured that one is composing C^{ν}-local diffeomorphisms [cf. Kupka and Sallet 1983, Stefan 1974b, Sussmann 1973]. We, however, can now do this for locally integrable time-dependence, considerably extending the scope of the theory expounded in the paper.

2. From flows to pseudogroups and from pseudogroups to Lie groupoids

In order to fully understand the relationship between our constructions and results for Lie groupoids with more common constructions and results in control theory, in this section we present the path from the latter to the former.

2.1. From flows to pseudogroups. Given a (possibly time-varying) vector field X on a manifold M, the flow of X defines, for each time $t_0 \in \mathbb{R}$ (the initial time), a family of local diffeomorphisms indexed by the final time t. In [Jafarpour and Lewis 2014] a comprehensive theory of time-varying vector fields is developed, providing the proper hypotheses for time-varying flows to depend on initial condition in a manner whose regularity agrees with the regularity of the vector field. Thus, the flow of a smooth vector field will define a family of smooth local diffeomorphisms and the flow of a real analytic vector field will define a family of real analytic diffeomorphisms. These families of local diffeomorphisms define a pseudogroup generated by these flows, as considered by Sussmann [1973]. By restricting to positive times we instead get a pseudosemigroup as considered by Kupka and Sallet [1983].

Let us make the preceding terminology and discussion precise. We note that the exact definition of pseudogroup tends to vary a little from author to author. Our definition is that which makes the connection with Lie groupoids most natural.

2.1 Definition: (Local diffeomorphisms, their inverses, and their compositions) Let $\nu \in \{\infty, \omega\}$ and let M be a C^ν-manifold.

(i) A ***$\mathbf{C}^{\boldsymbol{\nu}}$-local diffeomorphism*** of M is a mapping $\Phi\colon \mathcal{U} \to \mathsf{M}$, where $\mathcal{U} \subseteq \mathsf{M}$ is open and Φ is a C^ν-diffeomorphism onto its image.

(ii) The open set $\mathcal{U}$ is the ***domain*** of Φ and will sometimes be denoted by $\mathrm{dom}(\Phi)$. The open set $\Phi(\mathcal{U})$ is the ***range*** of Φ and will sometimes be denoted by $\mathrm{range}(\Phi)$.

(iii) The ***inverse*** of a local diffeomorphism Φ is $\Phi^{-1}\colon \mathrm{range}(\Phi) \to \mathsf{M}$.

(iv) If Φ_1 and Φ_2 are local diffeomorphisms, they are ***composable*** if $\mathrm{range}(\Phi_1) \cap \mathrm{dom}(\Phi_2) \neq \emptyset$, and their ***composition*** is the local diffeomorphism $\Phi_2 \circ \Phi_1$ with domain $\Phi_1^{-1}(\mathrm{dom}(\Phi_2))$ and range $\Phi_2(\mathrm{range}(\Phi_1))$.

The set of all C^ν-local diffeomorphisms of M is denoted by $\mathrm{Diff}^\nu_{\mathrm{loc}}(\mathsf{M})$. •

We may now define the notion of a pseudogroup.

2.2 Definition: (Multiplicative family of local diffeomorphisms, pseudogroup) Let $\nu \in \{\infty, \omega\}$ and let M be a C^ν-manifold. A ***$\mathbf{C}^{\boldsymbol{\nu}}$-multiplicative family*** on M is a family $\mathscr{M}$ of C^ν-local diffeomorphisms satisfying the following conditions:

(i) if $\Phi \in \mathscr{M}$ then $\Phi^{-1} \in \mathscr{M}$;

(ii) if $\Phi_1, \Phi_2 \in \mathscr{M}$ are composable, then $\Phi_2 \circ \Phi_1 \in \mathscr{M}$.

A C^ν-multiplicative family $\mathscr{M}$ is

(iii) ***restricting*** if $\Phi \in \mathscr{M}$ and if $\mathcal{V} \subseteq \mathrm{dom}(\Phi)$, then $\Phi|\mathcal{V} \in \mathscr{M}$, and is

(iv) ***localising*** if, given a local diffeomorphism Φ of M and an open cover $(\mathcal{U}_a)_{a\in A}$ of $\mathrm{dom}(\Phi)$ such that $\Phi|\mathcal{U}_a \in \mathscr{M}$ for each $a \in A$, we have $\Phi \in \mathscr{M}$.

(v) A ***$\mathbf{C}^{\boldsymbol{\nu}}$-prepseudogroup*** is a restricting C^ν-multiplicative family.

(vi) A ***$\mathbf{C}^{\boldsymbol{\nu}}$-pseudogroup*** is a restricting, localising C^ν-multiplicative family. •

In most applications of pseudogroups (or multiplicative families of local diffeomorphisms), one does not explicitly define the pseudogroup, but rather a set of generators, in the sense of the following result.

2.3 Proposition: (Multiplicative families and pseudogroups generated by a family of local diffeomorphisms) *Let $\nu \in \{\infty, \omega\}$, let M be a C^ν-manifold, and let $\mathbf{\Phi} = (\Phi_a)_{a \in A}$ be a family of C^ν-local diffeomorphisms of M such that $\mathsf{M} = \cup_{a \in A} \operatorname{dom}(\Phi_a)$. Then*

(i) there exists a smallest C^ν-multiplicative family $\mathscr{M}(\mathbf{\Phi})$ such that $\mathbf{\Phi} \subseteq \mathscr{M}(\mathbf{\Phi})$;

(ii) there exists a smallest C^ν-prepseudogroup $\mathscr{M}_{\mathrm{rstr}}(\mathbf{\Phi})$ such that $\mathbf{\Phi} \subseteq \mathscr{M}_{\mathrm{rstr}}(\mathbf{\Phi})$;

(iii) there exists a smallest C^ν-pseudogroup $\mathscr{P}(\mathbf{\Phi})$ such that $\mathbf{\Phi} \subseteq \mathscr{P}(\mathbf{\Phi})$.

Proof: In each part of the proof, we merely define the relevant object, leaving to the reader the relatively straightforward task of verifying that it has the desired properties.

(i) Let $\mathscr{M}(\mathbf{\Phi})$ be the set of C^ν-local diffeomorphisms of the form $\Phi_1^{\epsilon_1} \circ \dots \circ \Phi_k^{\epsilon_k}$, where $\Phi_j \in \mathbf{\Phi}$ and $\epsilon_j \in \{-1, 1\}$, $j \in \{1, \dots, k\}$, are such that $\Phi_1^{\epsilon_1} \circ \dots \circ \Phi_j^{\epsilon_j}$ and $\Phi_{j+1}^{\epsilon_{j+1}}$ are composable for $j \in \{1, \dots, k-1\}$.

(ii) We let

$$\mathscr{M}_{\mathrm{rstr}}(\mathbf{\Phi}) = \{\Psi|\mathcal{U} \mid \Psi \in \mathscr{M}(\mathbf{\Phi}),\ \mathcal{U} \subseteq \operatorname{dom}(\Psi)\}.$$

(iii) Let us define $\mathscr{P}(\mathbf{\Phi})$ to be the set of C^ν-local diffeomorphisms described as follows: $\Phi \in \mathscr{P}(\mathbf{\Phi})$ if, for every $x \in \operatorname{dom}(\Phi)$, there exists $a_1, \dots, a_k \in A$, $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$, and a neighbourhood $\mathcal{V} \subseteq \operatorname{dom}(\Phi)$ of x such that $\Phi_{a_1} \circ \dots \circ \Phi_{a_j}$ and $\Phi_{a_{j+1}}$ are composable for $j \in \{1, \dots, k-1\}$ and

$$\Phi|\mathcal{V} = \Phi_{a_1}^{\epsilon_1} \circ \dots \circ \Phi_{a_k}^{\epsilon_k}|\mathcal{V}. \qquad \blacksquare$$

2.4 Definition: (Multiplicative families generated by a family of local diffeomorphisms) Let $\nu \in \{\infty, \omega\}$ and let M be a C^ν-manifold.

(i) The C^ν-multiplicative family $\mathscr{M}(\mathbf{\Phi})$ constructed in the proof of Proposition 2.3(i) is the ***$\mathbf{C}^\nu$-multiplicative family generated by $\mathbf{\Phi}$***.

(ii) The C^ν-prepseudogroup $\mathscr{M}_{\mathrm{rstr}}(\mathbf{\Phi})$ constructed in the proof of Proposition 2.3(ii) is the ***$\mathbf{C}^\nu$-prepseudogroup generated by $\mathbf{\Phi}$***.

(iii) The C^ν-pseudogroup $\mathscr{P}(\mathbf{\Phi})$ constructed in the proof of Proposition 2.3(iii) is the ***$\mathbf{C}^\nu$-pseudogroup generated by $\mathbf{\Phi}$***. •

A special case that often arises is when one generates a pseudogroup from a restricting multiplicative family.

2.5 Definition: (Pseudogroup generated by a prepseudogroup) Let $\nu \in \{\infty, \omega\}$, let M be a C^ν-manifold, and let $\mathscr{P}$ be a C^ν-prepseudogroup. The ***$\mathbf{C}^\nu$-pseudogroup associated to $\mathscr{P}$*** is the C^ν-pseudogroup generated by $\mathscr{P}$, and is denoted by $\mathscr{P}^+$. •

Let us see how flows of vector fields give rise to pseudogroups.

2.6 Definition: (Pseudogroup associated to a vector field) Let $\nu \in \{\infty, \omega\}$, let M be a C^ν-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$.

(i) For $t_0, t \in \mathbb{T}$, let Φ_{t,t_0}^X be the C^ν-local diffeomorphism with domain

$$\operatorname{dom}(\Phi_{t,t_0}^X) = \{x \in \mathsf{M} \mid (t, t_0, x) \in D_X\}.$$

(ii) For $t_0 \in \mathbb{T}$, the ***(X, t_0)-pseudogroup*** is the C^ν-pseudogroup $\mathscr{P}(X, t_0)$ generated by the family Φ_{t,t_0}^X, $t \in \mathbb{T}$, of local diffeomorphisms. •

We observe that $\mathsf{M} = \cup_{t \in \mathbb{T}} \operatorname{dom}(\Phi_{t,t_0}^X)$, so the (X, t_0)-pseudogroup is well-defined.

2.2. From pseudogroups to Lie groupoids. Now we generalise from pseudogroups in the preceding section to Lie groupoids. This generalisation will come in handy, even when just dealing with Lie groupoids arising from pseudogroups, for defining what is meant by a "control system."

Let us first define what we mean by a groupoid and then a Lie groupoid; we refer to [e.g., Mackenzie 1987] for more details.

2.7 Definition: (Groupoid) Let B be a set. A ***groupoid*** over B is a set G such that, for every $x, y \in \mathsf{B}$, there exists a (possibly empty) subset $\mathsf{G}(x, y) \subseteq \mathsf{G}$ of ***arrows*** from x to y with the property that

$$\mathsf{G} = \cup\{\mathsf{G}(x, y) \mid x, y \in \mathsf{B}\},$$

along with the following mappings:

(i) for each $x \in \mathsf{B}$, an element $\mathrm{id}_x \in \mathsf{G}(x, x)$;

(ii) for each $x, y, z \in \mathsf{B}$, a mapping $\mathrm{comp}_{x,y,z} \colon \mathsf{G}(x, y) \times \mathsf{G}(y, z) \to \mathsf{G}(x, z)$, and we denote $\mathrm{comp}_{x,y,z}(g, h) = h * g$:

(iii) for each $x, y \in \mathsf{B}$, a mapping $\mathrm{inv}_{x,y} \colon \mathsf{G}(x, y) \to \mathsf{G}(y, x)$ and we denote $\mathrm{inv}_{x,y}(g) = g^{-1}$,

and these mappings must obey the following rules:

(iv) $\mathrm{id}_x * g = g$ and $h * \mathrm{id}_x = h$ for all $g \in \mathsf{G}(y, x)$ and $h \in \mathsf{G}(x, y)$, $x, y \in \mathsf{B}$;

(v) $h * (g * f) = (h * g) * f$ for all $f \in \mathsf{G}(x, y)$, $g \in \mathsf{G}(y, z)$, $h \in \mathsf{G}(z, w)$, $x, y, z, w \in \mathsf{B}$;

(vi) $g * g^{-1} = \mathrm{id}_y$ and $g^{-1} * g = \mathrm{id}_x$ for all $g \in \mathsf{G}(x, y)$, $x, y \in \mathsf{B}$.

A groupoid will often be denoted by $\mathsf{G} \rightrightarrows \mathsf{B}$. We have ***source*** and ***target*** mappings $\mathrm{src} \colon \mathsf{G} \to \mathsf{B}$ and $\mathrm{tgt} \colon \mathsf{G} \to \mathsf{B}$ defined by asking that $\mathrm{src}(g) = x$ and $\mathrm{tgt}(g) = y$ for $g \in \mathsf{G}(x, y)$. We denote

$$\mathsf{G}_2 = \{(g, h) \in \mathsf{G} \times \mathsf{G} \mid g \in \mathsf{G}(x, y),\ h \in \mathsf{G}(y, z) \text{ for some } x, y, z \in \mathsf{B}\}$$

and define $\mathrm{comp} \colon \mathsf{G}_2 \to \mathsf{G}$ by $\mathrm{comp}(g, h) = \mathrm{comp}_{x,y,z}(g, h)$ if $g \in \mathsf{G}(x, y)$ and $h \in \mathsf{G}(y, z)$. We also define $\mathrm{inv} \colon \mathsf{G} \to \mathsf{G}$ by $\mathrm{inv}(g) = \mathrm{inv}_{x,y}(g)$ if $g \in \mathsf{G}(x, y)$.

We also have the following constructions.

(vii) For $x \in \mathsf{B}$, the ***source fibre*** of x is $\mathsf{G}_x = \mathrm{src}^{-1}(x)$ and the ***target fibre*** of x is $\mathsf{G}^x = \mathrm{tgt}^{-1}(x)$. We also denote $\mathsf{G}_x^y = \mathsf{G}_x \cap \mathsf{G}^y = \mathsf{G}(x, y)$. •

We shall not deal too much with any sort of general groupoid, but mainly consider Lie groupoids whose definition we give shortly. We will, however, use the ***inverse groupoid*** of a groupoid $\mathsf{G} \rightrightarrows \mathsf{B}$ which is the groupoid $\mathsf{G}' \rightrightarrows \mathsf{B}$ defined by $\mathsf{G}' = \mathsf{G}$, $\mathsf{G}'(x, y) = \mathsf{G}(y, x)$, and with groupoid mappings $\mathrm{id}'_x = \mathrm{id}_x$, $\mathrm{comp}'_{x,y,z}(g, h) = \mathrm{comp}_{z,y,x}(h^{-1}, g^{-1})$, and $\mathrm{inv}'_{x,y} = \mathrm{inv}^{-1}_{x,y}$. Note that $\mathrm{src}'(g) = \mathrm{tgt}(g)$ and $\mathrm{tgt}'(g) = \mathrm{src}(g)$.

Let us now turn to Lie groupoids.

2.8 Definition: (Lie groupoid) Let $\nu \in \{\infty, \omega\}$. A groupoid $\mathsf{G} \rightrightarrows \mathsf{M}$ is a $\mathbf{C}^\nu$***-Lie groupoid*** if

(i) both G and M are C^ν-manifolds, with M Hausdorff and second countable, but G not necessarily so,

(ii) the groupoid mappings $x \mapsto \mathrm{id}_x$, comp, and $(x, y) \mapsto \mathrm{inv}_{x,y}$ are smooth, and

(iii) src and tgt are smooth submersions. •

Note that, by the Inverse and Implicit Function Theorems, the inverse groupoid of a Lie groupoid is a Lie groupoid.

Let us see how pseudogroups give rise to groupoids. We let $\nu \in \{\infty, \omega\}$, let M be a C^ν-manifold, and let $\mathscr{P}$ be a C^ν-prepseudogroup on M. Let $x \in \mathsf{M}$ and let $\mathscr{N}_x$ be the set of neighbourhoods of x. Let $\Phi, \Psi \in \mathscr{P}$ be such that $\mathrm{dom}(\Phi), \mathrm{dom}(\Psi) \in \mathscr{N}_x$. We say that Φ and Ψ are ***equivalent*** if there exists a neighbourhood $\mathcal{W} \subseteq \mathrm{dom}(\Phi) \cap \mathrm{dom}(\Psi)$ of x such that $\Phi|\mathcal{W} = \Psi|\mathcal{W}$. The set of such equivalence classes we denote by $\mathscr{G}_{x,\mathscr{P}}$. The equivalence class of a local diffeomorphism we denote by $[\Phi]_x$. We also denote $\mathscr{G}_{\mathscr{P}} = \mathring{\cup}_{x\in\mathsf{M}} \mathscr{G}_{x,\mathscr{P}}$, which is the ***groupoid of germs*** of $\mathscr{P}$. We make this a groupoid by defining $\mathrm{src}, \mathrm{tgt} \colon \mathscr{G}_{\mathscr{P}} \to \mathsf{M}$ by

$$\mathrm{src}([\Phi]_x) = x, \qquad \mathrm{tgt}([\Phi]_x) = \Phi(x),$$

and by defining groupoid composition, inversion, and identity by

$$[\Psi]_{\Phi(x)} * [\Phi]_x = [\Psi \circ \Phi]_x, \qquad [\Phi]_x^{-1} = [\Phi^{-1}]_{\Phi(x)}, \qquad \mathrm{id}_x = [\mathrm{id}]_x.$$

We can make this a Lie groupoid by defining the ***étalé topology*** for $\mathscr{G}_{\mathscr{P}}$ to be that with the basis

$$\mathcal{B}(\Phi) = \{[\Phi]_x \mid\ x \in \mathrm{dom}(\Phi)\}, \qquad \Phi \in \mathscr{P}.$$

It is clear that src is a local homeomorphism when $\mathscr{G}_{\mathscr{P}}$ has the étalé topology, and this establishes a differentiable structure for $\mathscr{G}_{\mathscr{P}}$. This differentiable structure is not generally second countable, and is generally Hausdorff only when $\nu = \omega$. In any case, $\mathscr{G}_{\mathscr{P}}$ is a Lie groupoid. In particular, we have the groupoid $\mathscr{G}_{\mathrm{Diff}^\nu_{\mathrm{loc}}(\mathsf{M})}$ associated to the pseudogroup of all local C^ν-diffeomorphisms.

It is also possible to extract a pseudogroup from a certain sort of Lie groupoid, and we refer to [Moerdijk and Mrcun 2003, Page 138] for this construction. One condition on a Lie groupoid $\mathsf{G} \rightrightarrows \mathsf{M}$ for there to be a nice correspondence between it and a groupoid associated with a pseudogroup as above is that the groupoid be what is known as an ***étalé Lie groupoid***, meaning that src and tgt are local diffeomorphisms. This will happen, for example, if both G and B have a well-defined dimension and these dimensions are the same.

3. Control systems in étalé Lie groupoids

In this section we introduce the notion of a control system in an étalé Lie groupoid, and the associated notions of reachability for such systems. In Section 3.3 we give conditions equivalent to reachability for systems whose reachable sets satisfy what we call the "generalised LARC," since this condition generalises the property of reachable sets for smooth control systems satisfying the usual LARC. We close the section, and the paper, by connecting in Section 3.4 our Lie groupoid notions of reachability to the more common notions in control theory.

3.1. Definition. We begin with the definition.

3.1 Definition: (Control system in an étalé Lie groupoid) Let $\nu \in \{\infty, \omega\}$ and let $\mathsf{G} \rightrightarrows \mathsf{M}$ be an étalé C^ν-Lie groupoid. A ***control system*** in G is an open submanifold Σ. For $x \in \mathsf{M}$, we denote $\Sigma_x = \Sigma \cap \mathrm{src}^{-1}(x)$. •

Given a control system Σ in an étalé Lie groupoid $\mathsf{G} \rightrightarrows \mathsf{M}$, we define a control system in the inverse groupoid $\mathsf{G}' \rightrightarrows \mathsf{M}$ by

$$\Sigma^{-1} = \{g \in \mathsf{G} \mid\ g^{-1} \in \Sigma\}.$$

In Section 3.4 we shall see how ordinary control systems give rise to various control systems in groupoids, according to the preceding definition.

3.2. Reachability definitions. With the above notion of control system in an étalé Lie groupoid, it is easy to define the associated notions of reachability.

We first define the reachable set.

3.2 Definition: (Reachable set) Let $\nu \in \{\infty, \omega\}$, let $\mathsf{G} \rightrightarrows \mathsf{M}$ be an étalé C^ν-Lie groupoid, and let $\Sigma \subseteq \mathsf{G}$ be a control system in G. For $x \in \mathsf{M}$, the ***reachable set*** from x for Σ is

$$\mathcal{R}_\Sigma(x) = \{\mathrm{tgt}(g) \mid\ g \in \Sigma_x\}.$$ •

Readers familiar with groupoids will notice that the reachable set for Σ from x is a subset of the orbit of the groupoid through x.

We shall use the notation

$$\Sigma_x^{\mathrm{int}} = \{g \in \Sigma_x \mid\ \mathrm{tgt}(g) \in \mathrm{int}(\mathcal{R}_\Sigma(x))\}.$$

It is now easy to give some standard notions of reachability for control systems in étalé Lie groupoids.

3.3 Definition: (Accessibility and reachability) Let $\nu \in \{\infty, \omega\}$, let $\mathsf{G} \rightrightarrows \mathsf{M}$ be an étalé C^ν-Lie groupoid, and let $\Sigma \subseteq \mathsf{G}$ be a control system in G. The control system Σ is:

(i) ***accessible*** from x_0 if $\mathrm{int}(\mathcal{R}_\Sigma(x_0)) \neq \emptyset$;

(ii) ***locally accessible*** from x_0 if, for every neighbourhood $\mathcal{U}$ of x_0, $\mathrm{int}(\mathcal{R}_\Sigma(x_0) \cap \mathcal{U}) \neq \emptyset$;

(iii) ***reachable*** from x_0 if $\mathcal{R}_\Sigma(x_0) = \mathsf{M}$;

(iv) ***locally reachable*** from x_0 if $x_0 \in \mathrm{int}(\mathcal{R}_\Sigma(x_0))$. •

As is well-known in the controllability literature for "ordinary" control systems, while it may be comparatively easy to characterise local accessibility, the characterisation of local reachability for completely general systems is a difficult task. The following condition on reachable sets is one that will allow us to give useful characterisations of reachability.

3.4 Definition: (Generalised LARC) Let $\nu \in \{\infty, \omega\}$, let $\mathsf{G} \rightrightarrows \mathsf{M}$ be an étalé C^ν-Lie groupoid and let $\Sigma \subseteq \mathsf{G}$ be a control system in G. The system Σ satisfies the ***generalised Lie algebra rank condition***, or the ***generalised LARC***, at x_0 if there exists a neighbourhood $\mathcal{U}$ of x_0 such that both Σ and Σ^{-1} are locally accessible from x for every $x \in \mathcal{U}$. •

We call this "the generalised LARC" because this condition is satisfied by an ordinary control system if it satisfies the classical LARC at x_0. The following property of systems satisfying the generalised LARC will be of importance for us.

3.5 Lemma: (Property of systems satisfying the generalised LARC) *Let $\nu \in \{\infty, \omega\}$, let $\mathsf{G} \rightrightarrows \mathsf{M}$ be an étalé C^ν-Lie groupoid, and let $\Sigma \subseteq \mathsf{G}$ be a control system in G. If Σ satisfies the generalised LARC from x_0, then Σ is locally reachable from x_0 if and only if $\mathrm{cl}(\mathcal{R}_\Sigma(x_0))$ contains a neighbourhood of x_0.*

Proof: Suppose that Σ is not locally reachable from x_0 and that $\mathrm{cl}(\mathcal{R}_\Sigma(x_0))$ contains a neighbourhood of x_0. Since Σ satisfies the generalised LARC, there is a neighbourhood $\mathcal{U}$ of x_0 such that $\mathcal{U} \subseteq \mathrm{cl}(\mathcal{R}_\Sigma(x_0))$ and such that Σ and Σ^{-1} are locally reachable from each $x \in \mathcal{U}$. Since Σ is not locally reachable from x_0, let $x \in \mathcal{U}$ be such that $x \notin \mathcal{R}_\Sigma(x_0)$. Since Σ^{-1} is locally accessible from x, $\mathrm{int}(\mathcal{R}_{\Sigma^{-1}}(x)) \cap \mathcal{U} \neq \emptyset$. Moreover, since $\mathcal{U} \subseteq \mathrm{cl}(\mathcal{R}_\Sigma(x_0))$, we have

$$\mathrm{int}(\mathcal{R}_{\Sigma^{-1}}(x)) \cap \mathcal{U} \cap \mathcal{R}_\Sigma(x_0) \neq \emptyset.$$

Thus we let $y \in \mathrm{int}(\mathcal{R}_{\Sigma^{-1}}(x)) \cap \mathcal{U} \cap \mathcal{R}_\Sigma(x_0)$, $g \in \Sigma_x^{-1}$, and $h \in \Sigma_{x_0}$ be such that $\mathrm{tgt}'(g) = y$ and $\mathrm{tgt}(h) = y$. Note that $g^{-1} \in \Sigma$ and that $\mathrm{src}(g^{-1}) = \mathrm{tgt}'(g) = y$. Thus the composition $g^{-1} * h$ is well-defined (in G) and

$$\mathrm{src}(g^{-1} * h) = \mathrm{src}(h) = x_0, \quad \mathrm{tgt}(g^{-1} * h) = \mathrm{tgt}(g^{-1}) = \mathrm{src}'(g) = x.$$

Thus $x \in \mathcal{R}_\Sigma(x_0)$, which contradiction gives this part of the result.

Conversely, if $\mathrm{cl}(\mathcal{R}_\Sigma(x_0))$ does not contain a neighbourhood of x_0, then clearly Σ is not locally reachable from x_0. ■

3.3. Conditions equivalent to local reachability. In this section we give conditions equivalent to local reachability for control systems in groupoids. To state the conditions, we make a few preliminary observations. If $g \in \mathsf{G}$, then there is a neighbourhood $\mathcal{U}$ of $\mathrm{src}(g)$ and a neighbourhood $\mathcal{V}$ of g in G such that $\mathrm{src}|\mathcal{V}$ is a C^ν-diffeomorphism onto $\mathcal{U}$ and $\mathrm{tgt}|\mathcal{V}$ is a C^ν-diffeomorphism onto its image. If $g \in \Sigma$, we can moreover choose $\mathcal{U}$ so that $\mathcal{V} \subseteq \Sigma$. Thus we have a local C^ν-diffeomorphism $(\Phi_g, \mathcal{U})$ defined by $\Phi_g = \mathrm{tgt} \circ (\mathrm{src}|\mathcal{V})^{-1}$. This then gives a germ of a local diffeomorphism at x that we denote by $\gamma_g \in \mathscr{G}_{x,\mathrm{Diff}^\nu_{\mathrm{loc}}(\mathsf{M})}$. For $x \in \mathsf{M}$ and $g \in \mathrm{src}^{-1}(x)$ we then have a mapping

$$\begin{aligned} \gamma_g^* \colon \mathrm{C}^\infty(\mathsf{M}) &\to \mathscr{C}^\infty_{x,\mathsf{M}} \\ f &\mapsto [\Phi_g^* f]_x \end{aligned}$$

(we note that the mapping is independent of the choice of representative Φ_g of the germ γ_g). We similarly define

$$\begin{aligned} j_\infty\gamma_g^* \colon \mathrm{C}^\infty(\mathsf{M}) &\to \mathsf{J}^\infty_x(\mathsf{M}; \mathbb{R}) \\ f &\mapsto j_\infty(\Phi_g^* f)(x). \end{aligned}$$

For $x \in \mathsf{M}$ we also define

$$\begin{aligned} \mathrm{germ}_x \colon \mathrm{C}^\infty(\mathsf{M}) &\to \mathscr{C}^\infty_{x,\mathsf{M}} \\ f &\mapsto [f]_x. \end{aligned}$$

All of the preceding mappings are $\mathbb{R}$-linear. Moreover, although we do not make use of this fact in this paper, all of the preceding mappings are continuous with appropriate topologies, which we now briefly describe. The topology for $\mathrm{C}^\infty(\mathsf{M})$ is the classical topology of uniform convergence of all derivatives on compact sets. One description of this topology is given by

Jafarpour and Lewis [2014, §3.1]. The topology for $\mathscr{C}^{\infty}_{x,\mathsf{M}}$ is the direct limit topology for the directed system $\mathrm{C}^{\infty}(\mathcal{U})$, $\mathcal{U}$ a neighbourhood of x. This is called the ***stalk topology*** and is discussed in some generality by Lewis [2014, §4.4]. The topology for $\mathsf{J}^{\infty}_{x}(\mathsf{M};\mathbb{R})$ is the inverse limit topology described in Section 1.2.3.

With this terminology, we have the following result.

3.6 Theorem: (Conditions equivalent to local reachability) *Let $\nu \in \{\infty, \omega\}$, let $\mathsf{G} \rightrightarrows \mathsf{M}$ be an étalé C^{ν}-Lie groupoid, and let $\Sigma \subseteq \mathsf{G}$ be a control system in G that satisfies the generalised LARC at x_0. Then the following statements are equivalent:*

(i) Σ *is locally reachable from* x_0*;*

(ii) $\cap_{g\in\Sigma_{x_0}} \ker(\mathrm{ev}_{x_0} \circ \gamma_g^*) \subseteq \ker(\mathrm{germ}_{x_0})$*;*

(iii) $\cap_{g\in\Sigma^{\mathrm{int}}_{x_0}} \ker(\gamma_g^*) \subseteq \ker(\mathrm{germ}_{x_0})$*;*

(iv) $\cap_{g\in\Sigma^{\mathrm{int}}_{x_0}} \ker(j_\infty\gamma_g^*) \subseteq \ker(\mathrm{germ}_{x_0})$*;*

(v) there exists no $f \in \mathrm{C}^{\infty}(\mathsf{M})$ with the following properties (thinking of f as a continuous linear functional on $\mathrm{C}^{\infty}(\mathsf{M})'$ in the weak-$*$ topology):

(a) $\mathrm{ev}_{x_0} \in \ker(f)$*;*

(b) for any neighbourhood $\mathcal{U}$ of x_0, we have

$$\{x \in \mathsf{M} \mid\ f(x) \in \mathbb{R}_{<0}\} \cap \mathcal{U} \neq \emptyset, \qquad \{x \in \mathsf{M} \mid\ f(x) \in \mathbb{R}_{>0}\} \cap \mathcal{U} \neq \emptyset;$$

(c) the hyperplane $\ker(f)$ *separates* $\mathrm{cl}(\mathcal{R}_\Sigma(x_0))$ *and* $(\mathsf{M} \setminus \mathrm{cl}(\mathcal{R}_\Sigma(x_0)))$.

Proof: (i) $\Longrightarrow$ (ii) Suppose that Σ is locally reachable from x_0. Thus there exists a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 such that

$$\mathcal{U} \subseteq \cup_{g\in\Sigma_{x_0}} \mathrm{tgt}(g).$$

Suppose that $f \in \mathrm{C}^{\infty}(\mathsf{M})$ satisfies $\mathrm{ev}_{x_0} \circ \gamma_g^*(f) = 0$ for every $g \in \Sigma_{x_0}$. Let $x \in \mathcal{U}$ and let $g_x \in \Sigma_{x_0}$ satisfy $x = \mathrm{tgt}(g_x)$. Let $\mathcal{U}' \subseteq \mathcal{U}$ be a neighbourhood of x_0 and $\mathcal{V}' \subseteq \mathsf{G}$ be a neighbourhood of g_x such that $\mathrm{src}|\mathcal{V}'$ is a diffeomorphism onto $\mathcal{U}'$. We then have

$$f(x) = f \circ \mathrm{tgt}(g_x) = f \circ \mathrm{tgt} \circ (\mathrm{src}|\mathcal{U}')^{-1}(x_0) = f \circ \Phi_g(x_0) = \mathrm{ev}_{x_0} \circ \gamma_{g_x}^*(f) = 0.$$

Then f is identically zero on $\mathcal{U}$ and so $f \in \ker(\mathrm{germ}_{x_0})$.

(ii) $\Longrightarrow$ (i) Suppose that Σ is not locally reachable. Since Σ is not locally reachable and satisfies the generalised LARC, Lemma 3.5 implies the following:

1. every neighbourhood of x_0 intersects $\mathcal{R}_\Sigma(x_0)$;
2. every neighbourhood of x_0 intersects $\mathcal{R}_\Sigma(x_0)^c \triangleq \mathsf{M} \setminus \mathrm{cl}(\mathcal{R}_\Sigma(x_0))$.

Therefore, there exists $f \in \mathrm{C}^{\infty}(\mathsf{M})$ with the property that $f(x) \in \mathbb{R}_{>0}$ for every $x \in \mathcal{R}_\Sigma(x_0)^c$ and $f(x) = 0$ for every $x \in \mathrm{cl}(\mathcal{R}_\Sigma(x_0))$ (this is a standard argument using bump functions and partitions of unity). For $g \in \Sigma_{x_0}$ we have

$$\mathrm{ev}_{x_0} \circ \gamma_g^*(f) = f \circ \mathrm{tgt}(g) = 0$$

since f vanishes on $\mathcal{R}_\Sigma(x_0)$. However, $\mathrm{germ}_{x_0}(f) \neq 0$ and this gives the desired conclusion.

(iii) $\Longrightarrow$ (ii) Suppose that (ii) does not hold. Thus there exists $f \in \mathrm{C}^\infty(\mathsf{M})$ such that $\mathrm{germ}_{x_0}(f) \neq 0$ but that $\mathrm{ev}_{x_0} \circ \gamma_g^*(f) = 0$ for all $g \in \Sigma_{x_0}$. Let $g \in \Sigma_{x_0}^{\mathrm{int}}$ and let $\mathcal{U} \subseteq \mathcal{R}_\Sigma(x_0)$ be a neighbourhood of $\mathrm{tgt}(g)$. Our hypotheses on f ensure that $f(x) = 0$ for all $x \in \mathcal{U}$. Let Φ_g be a representative of γ_g with the domain of Φ_g being some neighbourhood $\mathcal{V}$ of x_0. By shrinking $\mathcal{V}$, we can ensure that $\Phi_g(\mathcal{V}) \subseteq \mathcal{U}$. We then have $f \circ \Phi_g(x) = 0$ for all $x \in \mathcal{V}$, giving $\gamma_g^*(f) = 0$. This shows that (iii) does not hold.

(i) $\Longrightarrow$ (iv) Suppose that (i) holds and let

$$f \in \cap_{g \in \Sigma_{x_0}^{\mathrm{int}}} \ker(j_\infty \gamma_g^*).$$

This implies, in particular, that $f \circ \Phi_g(x_0) = 0$ for $g \in \Sigma_{x_0}^{\mathrm{int}}$. Since $x_0 \in \mathrm{int}(\mathcal{R}_\Sigma(x_0))$, this implies that $f(x) = 0$ for x in some neighbourhood of x_0. In other words, $\mathrm{germ}_{x_0}(f) = 0$.

(iv) $\Longrightarrow$ (iii) Let $f \in \cap_{\Sigma_{x_0}^{\mathrm{int}}} \ker(\gamma_g^*)$. This implies that, for each $g \in \Sigma_{x_0}^{\mathrm{int}}$, there is a neighbourhood $\mathcal{U}_g$ of x_0 such that $f \circ \Phi_g(x) = 0$ for $x \in \mathcal{U}_g$. This implies that the infinite jet of the righthand side is zero, and so $f \in \cap_{g \in \Sigma_{x_0}^{\mathrm{int}}} \ker(j_\infty \gamma_g^*)$. Thus $f \in \ker(\mathrm{germ}_{x_0})$, as desired.

(i) $\Longrightarrow$ (v) Suppose that there exists $f \in \mathrm{C}^\infty(\mathsf{M})$ with the three given properties. Consider the closed hyperplane $P_f = \ker(f)$ and the open half-spaces

$$H_f^+ = \{\alpha \in \mathrm{C}^\infty(\mathsf{M})' \mid \alpha(f) > 0\}, \quad H_f^- = \{\alpha \in \mathrm{C}^\infty(\mathsf{M})' \mid \alpha(f) < 0\}.$$

We suppose, without loss of generality, that $\mathsf{M} \backslash \mathrm{cl}(\mathcal{R}_\Sigma(x_0)) \subseteq H_f^-$. Let $\mathcal{U}$ be a neighbourhood of x_0. Condition (v b) on f ensures that $\mathcal{U} \cap H_f^- \neq \emptyset$. That is, every neighbourhood of x_0 contains points not in the closure of the reachable set, and so Σ is not locally reachable from x_0.

(v) $\Longrightarrow$ (i) Suppose that Σ is not locally reachable from x_0. Then, arguing as in the proof of the implication (ii) $\Longrightarrow$ (i) above, we have $f_-, f_+ \in \mathrm{C}^\infty(\mathsf{M})$ such that

1. $f_+(x) \in \mathbb{R}_{>0}$ for $x \in \mathrm{int}(\mathcal{R}_\Sigma(x_0))$,
2. $f_+(x) = 0$ for $x \in \mathsf{M} \setminus \mathrm{int}(\mathcal{R}_\Sigma(x_0))$,
3. $f_-(x) \in \mathbb{R}_{<0}$ for $x \in \mathcal{R}_\Sigma(x_0)^c$, and
4. $f_-(x) = 0$ for $x \in \mathrm{cl}(\mathcal{R}_\Sigma(x_0))$.

Then define $f \in \mathrm{C}^\infty(\mathsf{M})$ by $f = f_+ + f_-$. Therefore, since Σ is not locally reachable and satisfies the generalised LARC, by Lemma 3.5 f takes both positive and negative values in any neighbourhood of x_0. This ensures that condition (v b) holds. Since $x_0 \notin \mathrm{int}(\mathcal{R}_\Sigma(x_0))$ and since $x_0 \in \mathrm{cl}(\mathcal{R}_\Sigma(x_0))$, we have $f(x_0) = 0$ and so $\mathrm{ev}_{x_0} \in \ker(f)$. Finally, we have that $f(x) \geq 0$ for $x \in \mathrm{cl}(\mathcal{R}_\Sigma(x_0))$ and $f(x) < 0$ for $x \in \mathcal{R}_\Sigma(x_0)^c$. This is the desired conclusion. ■

The middle three conditions of the theorem all admit an interesting interpretation in terms of a generalisation of the usual notion of a point separating family of mappings.

3.7 Definition: (A_0-point separating) Let U, V, and W be $\mathbb{R}$-vector spaces, let $\mathscr{S} \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{V})$ be a family of linear maps, and let $A_0 \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{U}; \mathsf{W})$. The family $\mathscr{S}$ is

(i) ***A_0-point separating*** if $A(u) = 0$ for all $A \in \mathscr{S}$ implies that $A_0(u) = 0$;

(ii) ***point separating*** if it is id_{U}-point separating. •

With this definition we immediately have the following equivalences for the conditions of Theorem 3.6:

1. Condition (ii) $\iff$ $\{\mathrm{ev}_{x_0} \circ \gamma_g^* \mid g \in \Sigma_{x_0}\}$ is germ_{x_0}-point separating;
2. Condition (iii) $\iff$ $\{\gamma_g^* \mid g \in \Sigma_{x_0}^{\mathrm{int}}\}$ is germ_{x_0}-point separating;
3. Condition (iv) $\iff$ $\{j_\infty \gamma_g^* \mid g \in \Sigma_{x_0}^{\mathrm{int}}\}$ is germ_{x_0}-point separating.

3.4. Connection to usual notions of reachability. Let us see how the above definition of a control system in an étalé Lie groupoid relates to the usual notion of a control system. We allow here a rather general notion of system, namely that of a "tautological control system," following Lewis [2014]; in particular we refer to this cited work for a comprehensive discussion of the relationship between this quite general class of systems and more normal types of control systems, e.g., those of the "$\dot{x} = F(x,u)$" form. We let $\nu \in \{\infty, \omega\}$, let M be a C^ν-manifold, and let $\mathscr{F}$ be a subpresheaf of sets of C^ν-vector fields on M. For $\mathcal{U} \subseteq \mathsf{M}$ open, the subpresheaf $\mathscr{F}$ defines a subset $\mathscr{F}(\mathcal{U})$ of C^ν-vector fields on $\mathcal{U}$. For an interval $\mathbb{T} \subseteq \mathbb{R}$ we denote

$$\mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U})) = \{X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{T}\mathcal{U}) \mid X_t \in \mathscr{F}(\mathcal{U})\}.$$

If $T \in \mathbb{R}_{>0}$ then, for $X \in \mathrm{LI}\Gamma^\nu([0,T]; \mathscr{F}(\mathcal{U}))$ we have the local diffeomorphism $\Phi_{T,0}^X$, possibly with empty domain. We then define

$$\Sigma_{\mathscr{F},T} = \{[\Phi]_x \in \mathscr{G}_{\mathrm{Diff}^\nu_{\mathrm{loc}}(\mathsf{M})} \mid x \in \mathrm{dom}(\Phi),\ \Phi = \Phi_{T,0}^X \\ \text{for some } X \in \mathrm{LI}\Gamma^\nu([0,T]; \mathscr{F}(\mathcal{U})) \text{ and some } \mathcal{U} \subseteq \mathsf{M} \text{ open}\}.$$

We also define

$$\Sigma_{\mathscr{F},\leq T} = \cup_{t \in [0,T]} \Sigma_{\mathscr{F},t}, \qquad \Sigma_{\mathscr{F}} = \cup_{t \in \mathbb{R}_{\geq 0}} \Sigma_{\mathscr{F},t}.$$

Thus we see that associated to the tautological control system $(\mathsf{M}, \mathscr{F})$ are three sorts of control systems, $\Sigma_{\mathscr{F},T}$, $\Sigma_{\mathscr{F},\leq T}$, and $\Sigma_{\mathscr{F}}$, in the étalé Lie groupoid $\mathscr{G}_{\mathrm{Diff}^\nu_{\mathrm{loc}}(\mathsf{M})}$.

One then has the following notions of reachability for tautological control systems, made using our notions of reachability for control systems in groupoids.

3.8 Definition: (Accessibility and reachability for tautological control systems) Let $\nu \in \{\infty, \omega\}$ and let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with the associated groupoid control systems $\Sigma_{\mathscr{F},T}$, $\Sigma_{\mathscr{F},\leq T}$, and $\Sigma_{\mathscr{F}}$, as above. For $x_0 \in \mathsf{M}$, the system $\mathfrak{G}$ is:

(i) ***accessible*** from x_0 if $\Sigma_{\mathscr{F}}$ is accessible from x_0;

(ii) ***strongly accessible*** from x_0 if $\Sigma_{\mathscr{F},T}$ is accessible from x_0 for every $T \in \mathbb{R}_{>0}$;

(iii) ***locally accessible*** from x_0 if $\Sigma_{\mathscr{F}}$ is locally accessible from x_0;

(iv) ***strongly locally accessible*** from x_0 if $\Sigma_{\mathscr{F},T}$ is locally accessible for every $T \in \mathbb{R}_{>0}$;

(v) ***small-time accessible*** from x_0 if there exists $T \in \mathbb{R}_{>0}$ such that $\Sigma_{\mathscr{F},\leq t}$ is accessible from x_0 for every $t \in (0,T]$;

(vi) ***small-time strongly accessible*** if there exists $T \in \mathbb{R}_{>0}$ such that $\Sigma_{\mathscr{F},t}$ is accessible for every $t \in (0,T]$;

(vii) ***small-time locally accessible*** from x_0 if there exists $T \in \mathbb{R}_{>0}$ such that $\Sigma_{\mathscr{F},\leq t}$ is locally accessible from x_0 for every $t \in (0,T]$;

(viii) ***small-time strongly locally accessible*** if there exists $T \in \mathbb{R}_{>0}$ such that $\Sigma_{\mathscr{F},t}$ is locally accessible for every $t \in (0,T]$;

(ix) ***reachable*** from x_0 if $\Sigma_{\mathscr{F}}$ is reachable from x_0;

(x) ***strongly reachable*** from x_0 if $\Sigma_{\mathscr{F},T}$ is reachable from x_0 for every $T \in \mathbb{R}_{>0}$;

(xi) ***locally reachable*** from x_0 if $\Sigma_{\mathscr{F}}$ is locally reachable from x_0;

(xii) ***strongly locally reachable*** from x_0 if $\Sigma_{\mathscr{F},T}$ is locally reachable from x_0 for every $T \in \mathbb{R}_{>0}$;

(xiii) ***small-time reachable*** from x_0 if there exists $T \in \mathbb{R}_{>0}$ such that $\Sigma_{\mathscr{F},\leq t}$ is reachable from x_0 for every $t \in (0, T]$;

(xiv) ***small-time strongly reachable*** from x_0 if there exists $T \in \mathbb{R}_{>0}$ such that $\Sigma_{\mathscr{F},t}$ is reachable from x_0 for every $t \in (0, T]$;

(xv) ***small-time locally reachable*** from x_0 if there exists $T \in \mathbb{R}_{>0}$ such that $\Sigma_{\mathscr{F},\leq t}$ is locally reachable from x_0 for every $t \in (0, T]$;

(xvi) ***small-time strongly locally reachable*** if there exists $T \in \mathbb{R}_{>0}$ such that $\Sigma_{\mathscr{F},t}$ is locally reachable for every $t \in (0, T]$. •

This long list of definitions makes it clear that any of the standard notions of reachability in control theory can be enveloped by our groupoid theory.

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