Stability and Control of Large-scale Nonlinear Networks

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SJ and Francesco Bullo. Synchronization of Kuramoto Oscillators via Cutset Projections. IEEE Transactions on Automatic Control, 2019.

SJ and Elizabeth Y. Huang and Kevin D. Smith and Francesco Bullo. Flow and Elastic Networks on the *n*-torus: Geometry, Analysis, and Computation. SIAM Review, accepted, 2021.

SJ and Pedro Cisneros-Velarde and Francesco Bullo. Weak and Semi-Contraction for Network Systems and Diffusively-Coupled Oscillators. IEEE Transactions on Automatic Control, accepted, 2021.

Introduction: Large-scale Nonlinear Networks

Power grids

Brain neural network

Transportation network

Nonlinearity:

- Multiple equilibria
- Transient stability
- Cluster synchronization

Large-scale:

- Stochastic
- Distributed

- *"... As [power] systems become more heavily loaded, nonlinearities play an increasingly important role in power system behavior ... "* [I. Hiskens,1995]
- *"... in Oahu, Hawaii, at least 800,000 micro-inverters interconnect photovoltaic panels to the grid... "* [IEEE Spectrum, 2015]

Presentation outline

Synchronization and multi-stability in power grids

- oscillator networks and power grids
- transition from synchrony to incoherency
- multiplicity of equilibria

Incremental stability framework for nonlinear networks

- review of contraction theory
- weakly-contracting systems
- semi-contracting systems

Phenomenon #1: Transition form sync to incoherency

Frequency synchronization:

- Frequency synchronization is crucial for functionality and operation of power grids.
- Power electronic devices are designed to work at 60 *Hz* in the US (50 *Hz* in Europe)
- Increase in the network supply/demand and failure of lines can cause transition to incoherency.
- Loss of frequency synchronization leads to blackouts.

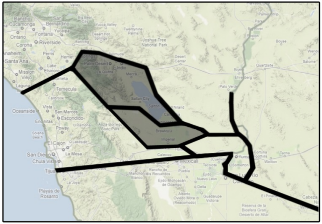
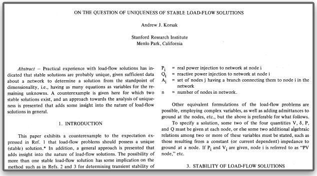


Figure: Southern California Blackout 2011– the gray area is the islanded part of the grid

Phenomenon #2: Multi-stable power flows

Theoretical observation:

- Multiple stable operating points exist in power networks
  [A. Korsak, On the question of uniqueness of stable load-flow solutions, 1972]



ON THE QUESTION OF UNIQUENESS OF STABLE LOAD-FLOW SOLUTIONS

Andrew J. Korsak

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Abstract – Practical experience with load-flow solutions has indicated that stable solutions are probably unique, given sufficient data about a network to determine a solution from the standpoint of dimensionality, i.e., having as many equations as variables for the remaining unknowns. A counterexample is given here for which two stable solutions exist, and an approach towards the analysis of uniqueness is presented that adds some insight into the nature of load-flow solutions in general.

1. INTRODUCTION

This paper exhibits a counterexample to the expectation expressed in Ref. 1 that load-flow problems should possess a unique (stable) solution.* In addition, a general approach is presented that adds insight into the nature of load-flow solutions. The possibility of more than one stable load-flow solution has some implication on the method such as in Refs. 2 and 3 for determining transient stability of

P_i = real power injection to network at node i
Q_i = reactive power injection to network at node i
A_i = set of nodes j having a branch connecting them to node i in the network
n = number of nodes in network.

Other equivalent formulations of the load-flow problems are possible, employing complex variables, as well as adding admittances to ground at the nodes, etc., but the above is preferable for what follows.

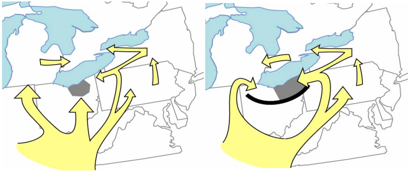
To specify a solution, some two of the four quantities V, δ, P, and Q must be given at each node, or else some two additional algebraic relations among two or more of these variables must be stated, such as those resulting from a constant (or current dependent) impedance to ground at a node. If P_i and V_i are given, node i is referred to as "PV node," etc.

3. STABILITY OF LOAD-FLOW SOLUTIONS

Practical observations:

1. Undesirable power flows around loops (Fig. (a): Lake Erie loop in the 2009 US blackout)
2. Dramatic change in power flow patterns before and after line trip (Fig. (b): Sammis–Star Trip in the 2003 US blackout)

(a)



(b)

Model: Coupled Oscillators Network

Pendulum clocks: "an odd kind of sympathy"
[C. Huygens, Horologium Oscillatorium, 1673]

Models for coupled oscillators:
[A. T. Winfree, 1967 and Y. Kuramoto 1975]

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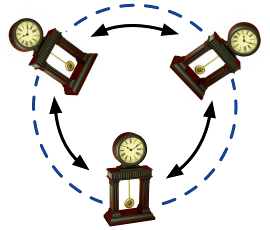
Kuramoto Model

1. **n-oscillators** with phases θ_i,
2. with natural frequencies $\omega_i \in \mathbb{R}$,
3. **coupling** with strength $a_{ij} = a_{ji}$.

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j).$$



Model: Active Power Dynamics

1. **generators** ■ and **inverters and loads** ●

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      active power p_i

Model: Active Power Dynamics

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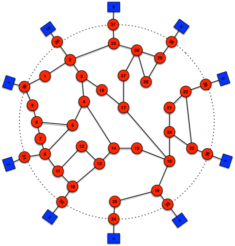
2. **physics:**
   1. Kirchhoff and Ohm laws
   2. **quasi-sync:** voltage and phase V_i, θ_i
      active power p_i
3. simplifying assumptions:
   1. lossless and inductive lines with admittances Y_{ij}
   2. decoupling of phase and voltage dynamics



New England IEEE 39-bus

Model: Active Power Dynamics

Structure-Preserving Model [A. Bergen & D. Hill, 1981]:

Active Power Dynamics

Generators: $M_i\ddot{\theta}_i + D_i\dot{\theta}_i = p_i - \sum_j a_{ij}\sin(\theta_i - \theta_j)$

Inverters: $\Lambda_i\dot{\theta}_i = p_i - \sum_j a_{ij}\sin(\theta_i - \theta_j)$

Loads: $\tau_i\dot{\theta}_i = p_i - \sum_j a_{ij}\sin(\theta_i - \theta_j)$

where

Active power capacity of line (i,j): $a_{ij} = |Y_{ij}|V_iV_j$

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where

Active power capacity of line (i,j): $\quad a_{ij} = |Y_{ij}|V_iV_j$

Synchronization= sync'd frequencies & bounded active power flows

$$\dot{\theta}_i = \omega_{\text{sync}},\ \forall \text{ bus } i \quad \& \quad |\theta_i - \theta_j| \le \gamma < \tfrac{\pi}{2},\ \forall \text{ line } (i,j)$$

Synchronization problem

Synchronization = Equilibrium point = Operating point

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \;\; \forall \text{ bus } i,$$

$$|\theta_i - \theta_j| \le \gamma < \frac{\pi}{2}, \;\; \forall \text{ line } (i, j)$$

Key questions

Given the network and the power profile:

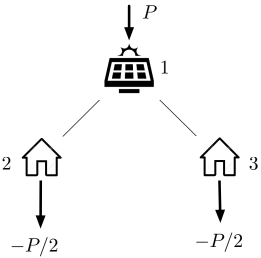
Q1: does there exist a **stable operating point**?

Q2: is the stable operating point **unique**?

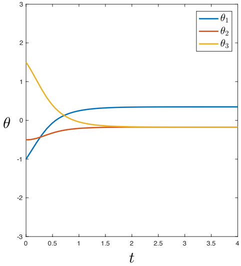
Q3: how to measure the **robustness** of the synchronization?

Q1: Existence of an operating point:

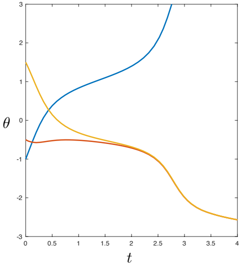
$$\dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



$a_{ij} = 1$



$P = 1$



$P = 2.5$

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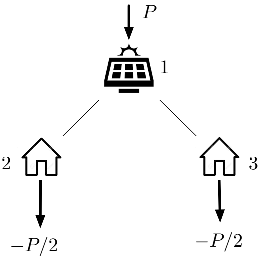

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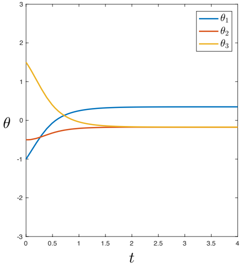
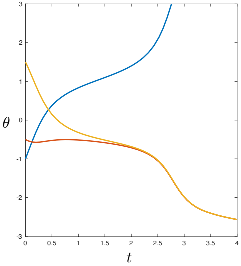


$P = 1$



$P = 2.5$

- sync threshold : "power transmission" vs. "coupling"
- quantify: "power transmission" $<$ "coupling"
- as a function of network parameters

Primer on algebraic graph theory

Weighted undirected graph with n nodes and m edges:

Incidence matrix: $n \times m$ matrix B s.t. $(B^\top p)_{(ij)} = p_i - p_j$

Edge weight matrix: $m \times m$ diagonal matrix $\mathcal{A}$

Laplacian matrix: $L = B\mathcal{A}B^\top$

Operating point:

$$p = B\mathcal{A}\sin(B^\top \theta)$$

Algebraic connectivity:

$$\lambda_2(L) = \text{second smallest eig of } L$$

$$= \text{notion of connectivity and coupling}$$

Known results

Given a network and p, does there exist angles?

$$p = B\mathcal{A}\sin(B^\top\theta),$$
$$|\theta_i - \theta_j| \leq \gamma < \frac{\pi}{2}, \ \ \forall \text{ line } (i,j).$$

synchronization arises if

power transmission $<$ **coupling strength**

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Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$$\|B^\top p\|_2 \quad < \sin(\gamma)\lambda_2(L) \qquad \text{for all graphs}$$

(Old 2-norm T)

(Old ∞-norm T)

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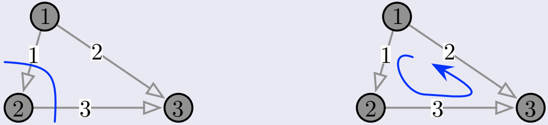
Equilibrium angles (neighbors within $\pi/2$ arc) exist if

$\Vert B^\top p\Vert_2$	$< \sin(\gamma)\lambda_2(L)$	for all graphs	(Old 2-norm T)
$\Vert B^\top L^\dagger p\Vert_\infty$	$< \sin(\gamma)$	for trees, complete	(Old ∞-norm T)

Novel: algebraic potential theory



$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\mathrm{Im}(B^\top)}_{\text{cutset space}} \oplus \underbrace{\mathrm{Ker}(B\mathcal{A})}_{\text{weighted cycle space}}$$

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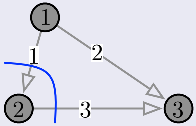

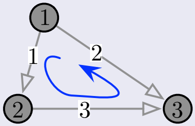

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$$\underbrace{\mathcal{P} = B^\top L^\dagger B \mathcal{A}}_{\text{cutset projection}} \quad = \text{oblique projection onto } \mathrm{Im}(B^\top)$$

parallel to $\mathrm{Ker}(B\mathcal{A})$

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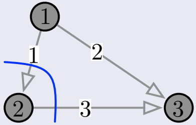

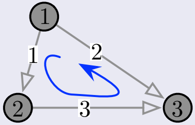

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parallel to $\mathrm{Ker}(B\mathcal{A})$

1. if G acyclic, then $\mathcal{P} = I_m$
2. if G unweighted, then $\mathcal{P}$ is an orthogonal projection
3. if $R_{\text{eff}} \in \mathbb{R}^{n\times n}$ are effective resistances, then $\mathcal{P} = -\frac{1}{2}B^\top R_{\text{eff}} B\mathcal{A}$

Rewriting the equilibrium equation

Find sufficient conditions on $B, \mathcal{A}, p$ s.t. there exists a solution θ to:

$$p = B\mathcal{A}\sin(B^\top\theta),$$
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Key idea: Node vs. Edge

$$p = B\mathcal{A}\sin(B^\top\theta) \qquad \textbf{Node balance eq. } \mathbb{R}^n$$
$$\Updownarrow$$
$$B^\top L^\dagger p = \mathcal{P}\sin(B^\top\theta) \qquad \textbf{Edge balance eq. } \mathbb{R}^m$$

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- **Edge variables**: $x = B^\top\theta$ and $z = B^\top L^\dagger p$

Find sufficient conditions on $z \in \mathrm{Im}(B^\top)$ s.t. there exists solution x to:

$$z = \mathcal{P}\sin(x) = \mathcal{P}[\mathrm{sinc}(x)]x$$

2. look for $x \in \mathcal{B}_q(\gamma) = \{x \mid \|x\|_q \leq \gamma\}$ solving

$$\mathcal{P}[\mathrm{sinc}(x)]x = z \quad \Longleftrightarrow \quad x = (\mathcal{P}[\mathrm{sinc}(x)])^{-1} z =: h(x)$$

Brouwer's Fixed-Point: A unifying theorem

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3. define min amplification factor of $\mathcal{P}[\mathrm{sinc}(x)] : \mathrm{Im}(B^\top) \to \mathrm{Im}(B^\top)$

$$\alpha_q(\gamma) := \min_{\|x\|_q \leq \gamma} \min_{\|y\|_q = 1} \|\mathcal{P}[\mathrm{sinc}(x)]y\|_q$$

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$$\|z\|_q \leq \gamma \alpha_q(\gamma) \quad \Longrightarrow \quad h \text{ satisfies Brouwer on } \mathcal{B}_q(\gamma)$$

Brouwer's Fixed-Point: A unifying theorem

Equilibrium angles (neighbors within γ arc) exist if, in some q-norm,

$$\|B^\top L^\dagger p\|_q \leq \gamma \alpha_q(\gamma) \qquad \text{for all graphs} \qquad \text{(New } q\text{-norm T)}$$

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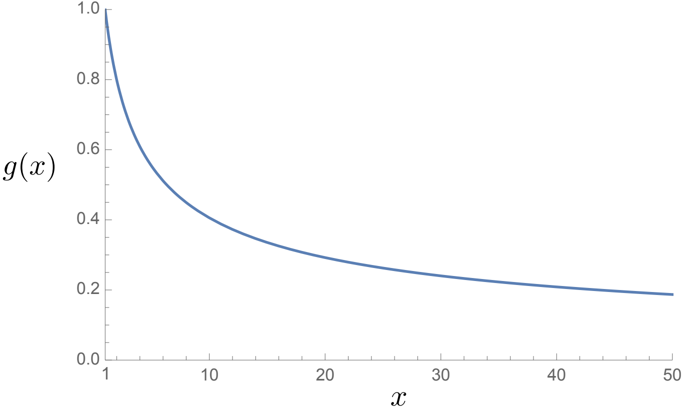
For $q = \infty$, the new test for all graphs

$$\|B^\top L^\dagger p\|_\infty \leq g(\|\mathcal{P}\|_\infty) \qquad \text{(New } \infty\text{-norm T)}$$

Function g is strictly decreasing



$g : [1, \infty) \to [0, 1]$

$$g(x) = \frac{y(x) + \sin(y(x))}{2} - x\frac{y(x) - \sin(y(x))}{2} \Big|_{y(x)=\arccos(\frac{x-1}{x+1})}$$

Comparison of synchronization tests

K_C = critical coupling of Kuramoto model, computed via MATLAB *fsolve* K_T = smallest value of scaling factor for which test T fails

Test Case	Critical ratio K_T/K_C			
	Old 2-norm	New ∞-norm $g(\lVert\mathcal{P}\rVert_\infty)$	Old ∞-norm Approx.test	New ∞-norm $\alpha_\infty(\pi/2)$
IEEE 9	16.54 %	73.74 %	92.13 %	85.06 %†
IEEE RTS 24	3.86 %	53.44 %	89.48 %	89.48 %†
New England 39-bus	2.97 %	67.57 %	100 %	100 %†
IEEE 118	0.29 %	43.70 %	85.95 %	—*
IEEE 300	0.20 %	40.33 %	99.80 %	—*
Polish 2383	0.11 %	29.08 %	82.85 %	—*

† *fmincon* has been run for 100 randomized initial phase angles.
* *fmincon* does not converge.

Old 2-norm:
$\lVert B^\top p\rVert_2 \leq \sin(\gamma)\lambda_2(L)$

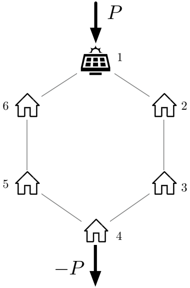
Old ∞-norm Approx.:
$\lVert B^\top L^\dagger p\rVert_\infty \leq \sin(\gamma)$

New ∞-norm:
$\lVert B^\top L^\dagger p\rVert_\infty \leq g(\lVert\mathcal{P}\rVert_\infty)$

New ∞-norm:
$\lVert B^\top L^\dagger p\rVert_\infty \leq \frac{\pi}{2}\alpha_\infty(\frac{\pi}{2})$

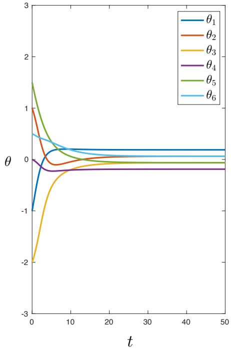
Q2: Is the operating point unique?

$$\dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

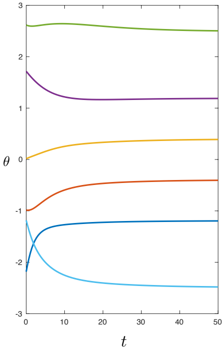


$a_{ij} = 1$

$P = 1/4$



$\theta_0 = [-1, 1, -2, 0, 1.5, 0.5]^\top$



$\theta_0 = [0, 1.2, 2.2, 3.9, 4.8, 1]^\top$

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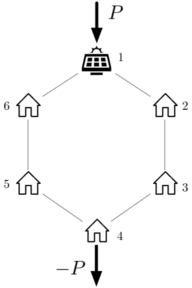
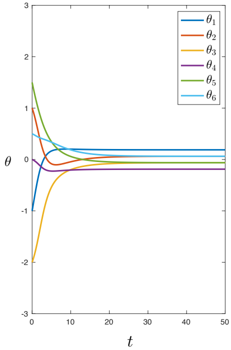
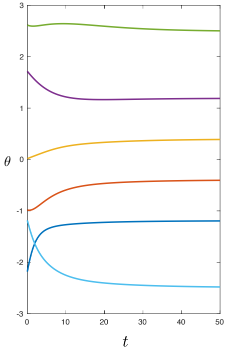


$a_{ij} = 1$

$P = 1/4$



$\theta_0 = [-1, 1, -2, 0, 1.5, 0.5]^\top$



$\theta_0 = [0, 1.2, 2.2, 3.9, 4.8, 1]^\top$

- multi-stable sync : “cycle structure” and “state space”
- quantify: “cycle structure” vs “multi-stable sync”

Key question

How to localize stable operating points?

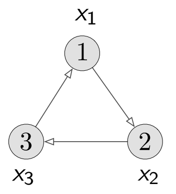
Winding number

Key question

How to localize stable operating points?

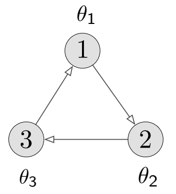
Winding number

Nodal variables in $\mathbb{R}^3$



$$\sum_{i=1}^{3} \overbrace{(x_i - x_{i+1})}^{\text{distance in } \mathbb{R}} = 0.$$

Nodal variables in $\mathbb{T}^3$



$$\sum_{i=1}^{3} \overbrace{(\theta_i - \theta_{i+1})}^{\text{distance in } \mathbb{S}} = 2\pi w_\sigma(\theta),$$

$w_\sigma(\theta) \in \mathbb{Z}$, winding number

Winding vector

Given a graph G with a cycle basis $\Sigma = \{\sigma_1, \dots, \sigma_{m-n+1}\}$ and $\theta \in \mathbb{T}^n$.

$$\text{Winding vector:} \qquad \mathbf{w}_\Sigma(\theta) = [w_{\sigma_1}(\theta), \dots, w_{\sigma_{m-n+1}}(\theta)]^\top$$

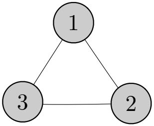
Theorem: Kirchhoff angle law on $\mathbb{T}^n$

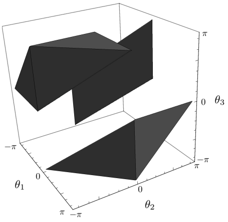
$$\begin{aligned} w_\sigma(\theta) = 0, \pm 1, \dots, \pm \lfloor n_\sigma/2 \rfloor \\ \Longrightarrow \quad \mathbf{w}_\Sigma(\theta) \quad &\text{is piecewise constant,} \\ \mathbf{w}_\Sigma(\theta) \quad &\text{takes value in a finite set} \end{aligned}$$

Winding cells: equivalence classes

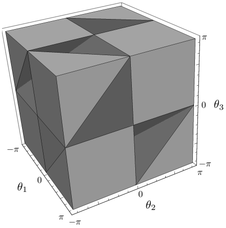
Given a graph G with a cycle basis Σ. For every $\mathbf{u} \in \mathbb{Z}^{m-n+1}$

$$\text{(Winding cell } \mathbf{u}) \quad = \quad \text{all } \theta \in \mathbb{T}^n \text{ s.t. } \mathbf{w}_\Sigma(\theta) = \mathbf{u}.$$

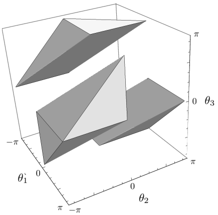




$\mathbf{u} = -1$



$\mathbf{u} = 0$

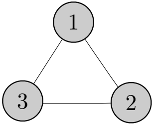


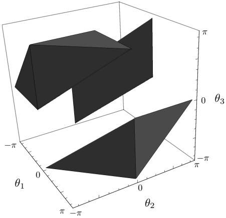
$\mathbf{u} = +1$

Theorem: Winding partition of n-torus

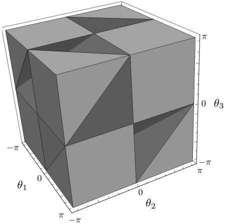
$$\mathbb{T}^n = \bigcup_{\mathbf{u} \in \mathbb{Z}^{m-n+1}} \text{(Winding cell } \mathbf{u})$$

Properties of the winding partition

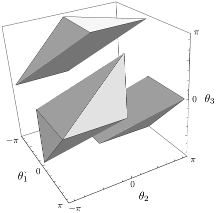




$\mathbf{u} = -1$

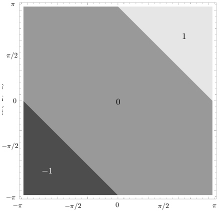


$\mathbf{u} = 0$



$\mathbf{u} = +1$

- each winding cell is connected
- each winding cell is invariant under rotation
- **bijection**: winding cell $\longleftrightarrow$ convex polytope



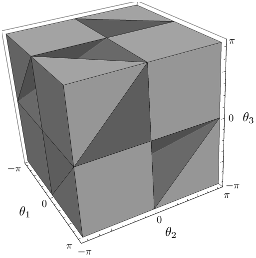
At-most uniqueness in winding cells

Multi-stability

Given topology, admittances, and injections

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \;\; \forall \text{ bus } i$$

$$|\theta_i - \theta_j| \le \gamma < \frac{\pi}{2}, \;\; \forall \text{ line } (i,j)$$



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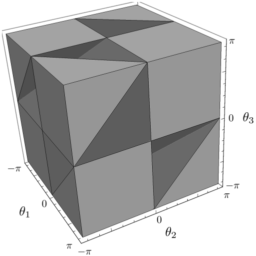


Theorem: At-most-uniqueness and extensions

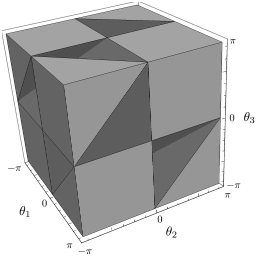
At-most uniqueness in winding cells

Multi-stability

Given topology, admittances, and injections

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \quad \forall \text{ bus } i$$

$$|\theta_i - \theta_j| \le \gamma < \frac{\pi}{2}, \quad \forall \text{ line } (i,j)$$



Theorem: At-most-uniqueness and extensions

1. each winding cell has at-most-unique equilibrium with $\Delta\theta \le \gamma$

At-most uniqueness in winding cells

Multi-stability

Given topology, admittances, and injections

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \quad \forall \text{ bus } i$$

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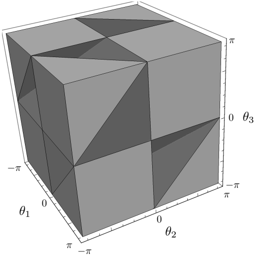


Theorem: At-most-uniqueness and extensions

1. each winding cell has at-most-unique equilibrium with $\Delta\theta \leq \gamma$
2. equilibrium loop flow increases monotonically wrt winding number

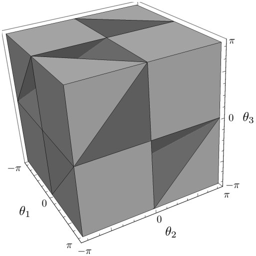
At-most uniqueness in winding cells

Multi-stability

Given topology, admittances, and injections

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \ \ \forall \text{ bus } i$$

$$|\theta_i - \theta_j| \le \gamma < \frac{\pi}{2}, \ \ \forall \text{ line } (i,j)$$



Theorem: At-most-uniqueness and extensions

1. each winding cell has at-most-unique equilibrium with $\Delta\theta \le \gamma$
2. equilibrium loop flow increases monotonically wrt winding number
3. existence of solutions in WindingCell(u) with $\Delta\theta \le \gamma$ if

$$\|B^\top L^\dagger p + C\mathbf{u}\|_\infty \le g(\|\mathcal{P}\|_\infty), \text{ or} \qquad \text{(Static Test)}$$

$$\eta^{k+1} = \eta^k - \alpha\mathcal{P}(\operatorname{asin}(\eta^k) - C\mathbf{u}), \qquad \text{(Iteration)}$$

Summary

- geometry of cutset projection operator
- family of sufficient sync conditions
- partiton of n-torus based on winding vector
- localize the equilibrium points using winding cells

Future research

- close the gap between sufficient and necessary conditions
  - use suitable optimization algorithms to compute min amplification factor.
- region of attraction of stable equilibrium points
  - design suitable Lyapunov functions which can ensure convergence with transient guarantees
- generalizations to other oscillator models.
  - extension to multi-stability of state-space coupled oscillators such as FitzHugh–Nagumo

Presentation outline

Synchronization and multi-stability in power grids

- oscillator networks and power grids
- transition from synchrony to incoherency
- multiplicity of equilibria

Incremental stability framework for nonlinear networks

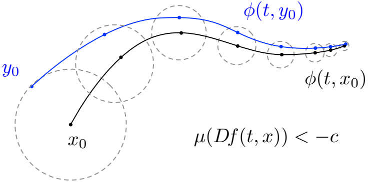
- review of contraction theory
- weakly-contracting systems
- semi-contracting systems

Definition: Contracting systems

$\dot{x} = f(t, x)$ is contracting wrt to $\|\cdot\|$ with rate $c > 0$:

$$\|x(t) - y(t)\| \leq e^{-ct}\|x(0) - y(0)\|.$$

Contracting system: flow is a contracting map.



- D. C. Lewis. Metric properties of differential equations. *American Journal of Mathematics*, 71(2):294–312, 1949
- B. P. Demidovich. Dissipativity of a nonlinear system of differential equations. *Uspekhi Matematicheskikh Nauk*, 16(3(99)):216, 1961
- W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998

Highly ordered asymptotic and transient behaviors:

1. initial conditions are forgotten
2. time-invariant f: unique globally stable equilibrium
3. periodic f: unique globally stable periodic solution
4. robustness properties: input-to-state stability
5. finite input-state gain in the presence of (Lipschitz continuous) unmodeled dynamics.

Combines in unified coherent framework results from:

1. stability notions and Lyapunov functions
2. Banach contraction and Brouwer fixed point theorems,
3. monotone systems theory, and
4. geometry of Banach, Riemannian and Finsler spaces

Contraction theory includes results about (1) the existence of equilibria (not just their stability), (2) entrainment to periodic solutions, and (3) the dichotomy for weakly contracting systems

Matrix measures and contracting systems

Definition: matrix measure

The **matrix measure** of $A \in \mathbb{R}^{n \times n}$ wrt to $\|\cdot\|$:

$$\mu_{\|\cdot\|}(A) := \lim_{h \to 0^+} \frac{\|I_n + hA\| - 1}{h}.$$

- Directional derivative of norm $\|\cdot\|$ in direction of A,
- One-sided Lipschitz constant:

E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I. Nonstiff Problems.*
Springer, 1993

Theorem: contraction via matrix measures

$\dot{x} = f(t,x)$ with f continuously differentiable in x is contracting wrt to $\|\cdot\|$ with rate $c > 0$ iff

$$\mu(Df(t,x)) \leq -c, \qquad \text{for every } t, x$$

Computing matrix measures

For $\|\cdot\|_{2,P}$ with $P \succ 0$:

$$\mu_{2,P}(Df(t,x)) \leq -c \iff PDf(t,x) + Df(t,x)^\top P \preceq -cP$$

For $\|\cdot\|_{1,[\eta]}$ with $\eta \in \mathbb{R}^n_{>0}$ and $Df(t,x)$ Metzler:

$$\mu_{1,[\eta]}(Df(t,x)) \leq -c \iff \eta^\top Df(t,x) \leq -c\eta^\top$$
$$\mu_{\infty,[\eta]^{-1}}(Df(t,x)) \leq -c \iff Df(t,x)\eta \leq -c\eta$$

If f is polynomial in t and x,

1. can be checked using SOS programming
2. search for the weights ($P \succ 0$ and $\eta \in \mathbb{R}^n_{>0}$) using SOS programming

   E. M. Aylward, P. A. Parrilo, and J.-J. E. Slotine. Stability and robustness analysis of nonlinear systems via contraction metrics and SOS programming. *Automatica*, 44(8):2163–2170, 2008.

Contraction theory for networks

Challenge: many real-world networks are not contracting.

Network flow system $\dot{x} = f(x)$ preserving commodity x:

$$\begin{aligned} \text{constant} &= \mathbb{1}_n^\top x(t) \\ &\implies 0 = \mathbb{1}_n^\top \dot{x}(t) = \mathbb{1}_n^\top f(x(t)) \\ &\implies \mathbb{0}_n = \mathbb{1}_n^\top Df(x(t)) \end{aligned}$$

If additionally f has Metzler Jacobian, then $\mu_1(Df(x)) = 0$.

Weakly-contracting systems

Definition: Weakly-contracting systems

$\dot{x} = f(t,x)$ with f continuously differentiable in x is weakly-contracting wrt $\|\cdot\|$:

$$\mu_{\|\cdot\|}(Df(t,x)) \leq 0$$

Theorem: Dichotomy for weakly-contracting systems

For a weakly-contracting system $\dot{x} = f(x)$, either

1. f has no equilibrium and every trajectory is unbounded, or
2. f has at least one equilibrium x^* and every trajectory is bounded, and:
   - (i) each equilibrium x^{**} is stable with weak Lyapunov function $x \mapsto \|x - x^{**}\|$,
   - (ii) if the norm $\|\cdot\|$ is a (p,R)-norm, $p \in \{1,\infty\}$ and f is piecewise real analytic, then every trajectory converges to the set of equilibria,
   - (iii) x^* is locally asy stable $\implies$ x^* is globally asy stable.

Examples of weakly-contracting networks

1. Lotka-Volterra population dynamics (Lotka, 1920; Volterra, 1928), (weakly-contracting wrt ℓ_1-norm)
2. Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981), (weakly-contracting wrt ℓ_1-norm and ℓ_∞-norm)
3. Daganzo's cell transmission model for traffic networks (Daganzo, 1994), (weakly-contracting wrt ℓ_1-norm)
4. compartmental systems in biology, medicine, and ecology (Sandberg, 1978; Maeda et al., 1978). (weakly-contracting wrt ℓ_1-norm)
5. saddle-point dynamics for optimization of weakly-convex functions (Arrow et al., 1958). (weakly-contracting wrt ℓ_2-norm)

Example: Distributed primal-dual algorithm over networks

Optimization problem

$$\min_{x \in \mathbb{R}^k} f(x) = \min_{x \in \mathbb{R}^k} \sum_{i=1}^{n} f_i(x)$$

Distributed implementation

- n agents communicate over a undirected weighted graph G,
- agent i have access to function f_i and can exchange x_i with its neighbors.

$$\min_{x \in \mathbb{R}^k} \quad \sum_{i=1}^{n} f_i(x_i)$$

$$x_1 = x_2 = \ldots = x_n$$

In matrix form by assuming $x = (x_1^\top, \ldots, x_n^\top)^\top \in \mathbb{R}^{nk}$:

$$\min_{x \in \mathbb{R}^k} \quad \sum_{i=1}^{n} f_i(x_i)$$

$$(L \otimes I_k)x = \mathbb{0}_{nN}$$

Example: Distributed primal-dual algorithm

If each f_i is continuously differentiable in x_i:

Distributed primal-dual algorithm (component form):

$$\dot{x}_i = -\nabla f_i(x_i) - \sum_{j=1}^{n} a_{ij}(\nu_i - \nu_j),$$
$$\dot{\nu}_i = \sum_{j=1}^{n} a_{ij}(x_i - x_j)$$

Distributed primal-dual algorithm (vector form):

$$\dot{x} = -\nabla f(x) - (L \otimes I_k)\nu,$$
$$\dot{\nu} = (L \otimes I_k)x$$

Example: Distributed primal-dual algorithm

Assume

1. f has a minimum $x^* \in \mathbb{R}^k$,
2. for each $i \in \{1, \dots, n\}$, f_i is twice differentiable, $\nabla^2 f_i(x) \succeq 0$ for all x, and $\nabla^2 f_i(x^*) \succ 0$, and
3. the undirected weighted graph G is connected with Laplacian L.

Theorem: Distributed primal-dual dynamics

The distributed primal-dual algorithm

1. is weakly-contracting wrt ℓ_2-norm,
2. its trajectory $(x(t), \nu(t))$ converges exponentially to $(\mathbb{1}_n \otimes x^*, \mathbb{1}_n \otimes \nu^*)$, where $\nu^* = \sum_{i=1}^n \nu_i(0)$,
3. its exponential convergence rate is $-\alpha_{\mathrm{ess}}\left(\begin{bmatrix} -\nabla^2 f(x^*) & -L \otimes I_k \\ L \otimes I_k & \mathbb{0} \end{bmatrix}\right)$ where

$$\alpha_{\mathrm{ess}}(A) := \max\{\Re(\lambda) \mid \lambda \in \mathrm{spec}(A) \setminus \{0\}\}.$$

Semi-contracting systems: matrix semi-measures

- Idea: flows converges to each other only in **certain directions**.

Definition: Semi-norms

$|||\cdot|||$ is a *semi-norm* if

1. $|||cv||| = |c||||v|||$, for every $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$;
2. $|||v + w||| \leq |||v||| + |||w|||$, for every $v, w \in \mathbb{R}^n$.

- Define the subspace $\mathrm{Ker}\,|||\cdot||| = \{v \in \mathbb{R}^n \mid |||v||| = 0\}$.
- Example: for $k < n$, $R \in \mathbb{R}^{k\times n}$, and norm $\|\cdot\|$, we get $|||x|||_R = \|Rx\|$.

Definition: Matrix semi-measures

The **matrix semi-measure** of $A \in \mathbb{R}^{n\times n}$ wrt $|||\cdot|||$:

$$\mu_{|||\cdot|||}(A) = \lim_{h\to 0^+} \frac{|||I_n + hA||| - 1}{h}.$$

- Directional derivative of $|||\cdot|||$ in direction of A.

Semi-contracting systems

Definition: Semi-contracting systems

$\dot{x} = f(t,x)$ with f continuously differentiable in x is semi-contracting wrt the semi-norm $|||\cdot|||$ with rate $c > 0$:

$$\mu_{|||\cdot|||}(Df(t,x)) \leq -c$$

Theorem: Semi-contracting systems

Consider $\dot{x} = f(t,x)$ with f continuously differentiable in x and assume

- f is semi-contracting wrt the semi-norm $|||\cdot|||$ with rate $c > 0$, and
- **(Infinitesimal invariance)**: $Df(t,x)\,\mathrm{Ker}\,|||\cdot||| \subseteq \mathrm{Ker}\,|||\cdot|||$, for every t, x

Then,

1. for every two trajectories $x(t), y(t)$,

$$|||x(t) - y(t)||| \leq e^{-ct}|||x(0) - y(0)|||, \qquad \text{for every } t \geq 0.$$

2. For time-invariant f, trajectories converge to an affine subspace $x^* + \mathrm{Ker}\,|||\cdot|||$.

Examples of semi-contracting networks

1. Kuramoto oscillators (Kuramoto, 1975) and coupled swing equations (Bergen and Hill, 1981), (semi-contracting wrt ℓ_1-norm)
2. Chua's diffusively-coupled circuits (Wu and Chua, 1995), (semi-contracting wrt ℓ_2-norm)
3. morphogenesis in developmental biology (Turing, 1952), (semi-contracting wrt ℓ_1-norm)
4. Goodwin model for oscillating auto-regulated gene (Goodwin, 1965). (semi-contracting wrt ℓ_1-norm)

Example: Diffusively-coupled oscillators

- n agents connected through a weighted undirected graph G with Laplacian L,
- internal dynamics of each agent is identical and given by $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^k \to \mathbb{R}^k$

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

Theorem: diffusively-coupled oscillators are semi-contracting

Suppose that

$$\mu_{p,Q}(Df(t,x)) \leq \lambda_2(L) - c, \qquad \text{for every } t, x$$

then

1. for every two trajectories $x(t)$ and $y(t)$,

$$\|x(t) - y(t)\|_{2,p,(R\otimes Q)} \leq e^{-ct} \|x(0) - y(0)\|_{2,p,(R\otimes Q)}.$$

2. the system achieves synchronization: $\lim_{t\to\infty} x(t) = \mathbb{1}_n \otimes x_{\mathrm{ave}}(t)$

Summary

- Review contraction theory and matrix measures
- Two extensions of classical contraction:
  - weak contraction
  - semi-contraction
- Properties of weakly-contracting and semi-contracting systems

Future research

- Contraction-based compositional analysis of interconnected systems
  - scalable stability certificates using non-Euclidean contraction.
- Optimization algorithms using contraction theory
  - extension to gradient descent algorithms and time-varying algorithms.
  - connection with discrete-time algorithms for optimization.
- Learning stable system from trajectory data
  - use contraction condition as side-information in the optimization problem.