Real analytic control systems*

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Abstract

Using a suitable locally convex topology for the space of real analytic vector fields, we give a characterization of real analytic control systems. Among other things, this class of real analytic control systems has the property that, upon substitution of an open-loop control, the resulting time-varying vector field has a flow depending on initial condition in a real analytic manner. To give context to the real analytic case, we also consider the cases of finitely differentiable and smooth control systems.

1. Introduction

In geometric control theory there has been for a long time an understanding that real analytic systems are distinguished from smooth systems [Sussmann 1990]. There are both physical and mathematical reasons for this. Physically, any smooth model one encounters will certainly also be real analytic, i.e., real analyticity, not smoothness, is what nature demands. Mathematically, real analyticity typically allows one to weaken hypotheses, e.g., hypotheses of finite generation [Agrachev and Sachkov 2004, Lemma 5.2], and/or strengthen conclusions, e.g., the tangent spaces to orbits in the real analytic orbit theorem [Nagano 1966, Sussmann 1974].

In this paper we address the problem of what is the "correct" definition of a real analytic control system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = F(u(t), x(t)), \tag{1.1}$$

where $\mathcal{U}$ is a topological space, $u : \mathbb{T} \to \mathcal{U}$ is an appropriate control, and $F : \mathcal{U} \times M \to TM$ is a parametrized vector field, i.e., for every $u \in \mathcal{U}$ and every $x \in M$, we have $F(u, x) \in T_xM$. In fact, we shall simultaneously develop the definitions in the finitely differentiable and smooth cases.

For $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, fundamental properties of solutions of (1.1), such as existence, uniqueness, and C^ν-dependence of solutions on initial conditions have been studied in literature. In studying these properties, the dependence of the parametrized vector field F on x and u plays a significant role. In considering these matters, one usually works with the time-varying vector field F^u that is obtained by substituting a (say, bounded measurable) open-loop control $t \mapsto u(t)$ in the parametrized vector field F. As is classically known, e.g., [Sontag 1998, Theorem 45], if F^u is measurable in t when x is fixed, Lipschitz in x when t is fixed, and if the Lipschitz constant is bounded by a locally integrable function

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of t, then there exists a unique solution to (1.1) and the solution is C^0 with respect to the initial condition. The idea of this proof can be extended to the finitely differentiable case. In [Agrachev and Sachkov 2004], it is stated that, if the time-varying vector field F^u is measurable in t when x is fixed, C^∞ in x when t is fixed, and all its partial derivatives with respect to x are locally integrable functions, then the solutions of (1.1) are C^∞ with respect to the initial conditions. In our presentation below we will give conditions on F that ensure that the preceding hypotheses for F^u are met for all bounded measurable controls.

Corresponding results are, until now, not known in the real analytic case. One can, however, find some stronger conditions in the literature to guarantee the C^ω-dependence of solutions on the initial condition. For instance, in [Sontag 1998, Proposition C.3.12] it is shown that joint C^ω with respect to x and t gives a flow that is C^ω with respect to initial condition. It bears pointing out that this is a strong condition, since it requires the dependence on time to be real analytic. This circumstance will not often arise in control theory.

In this paper, we present a unified approach to the study of regularity in control theoretic models in the case of finitely differentiable, smooth, and, most significantly, real analytic models. For the first time in the literature we provide a means of dealing with real analytic systems in a manner that (i) gives useful results such as real analytic dependence on initial conditions and (ii) provides tools for real analytic systems on a par with those available for finitely differentiable or smooth systems. The main idea is to consider a control system of the form (1.1) as a mapping from the control set to the the space of C^ν-vector fields, i.e., we consider the map $\widehat{F} : \mathcal{U} \to \Gamma^\nu(TM)$ defined by

$$\widehat{F}(u)(x) = F(u, x). \tag{1.2}$$

By introducing a family of seminorms, we define a topology on $\Gamma^\nu(TM)$. Using this topology, we find a joint regularity condition for C^ν-systems that ensures the C^ν-dependence of the solutions of (1.1) with respect to the initial condition.

2. Notation, conventions, and mathematical background

In this section we provide the mathematical background that is needed to read the paper. Although we state all the mathematical definitions and results rigorously, our treatment is far from comprehensive. We refer to [Jafarpour and Lewis 2013] for the comprehensive treatment.

In this paper, when we say "class C^ν" we assume that $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$. We shall write $\nu+1$ a few times, and this shall have the obvious meaning, i.e., normal addition when $\nu \in \mathbb{Z}_{\geq 0}$ and $\infty+1 = \infty$ and $\omega+1 = \omega$. Manifolds will be of class C^r, $r \in \{\infty, \omega\}$ and will be Hausdorff and second countable. We shall often use the terminology, "let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required." This has the more or less obvious meaning that $r = \omega$ if $\nu = \omega$ and $r = \infty$ otherwise. Manifolds of class C^∞ are called ***smooth manifolds*** and manifolds of class C^ω are called ***real analytic manifolds***.

Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold. The set of C^ν-vector fields is denoted by $\Gamma^\nu(TM)$. This is a $\mathbb{R}$-vector space with the usual pointwise operations of addition and scalar multiplication. If $X \in \Gamma^\nu(TM)$, then, for $m \in \mathbb{Z}_{\geq 0}$ satisfying $m \leq \nu$, the m-jet of X at a point $x \in M$ is denoted by $j_m X(x)$. The set of m-jets of all C^ν-vector fields is denoted by J^mTM. One can give J^mTM a natural C^r-vector bundle structure where the projection map π_m is defined by

$$\begin{aligned}\pi_m : J^mTM &\to M,\\ j_mX(x) &\mapsto x\end{aligned}$$

[Kolář, Michor, and Slovák 1993, §12.17].

Let $\mathcal{X}$ be a topological space and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A function $f : \mathbb{T} \to \mathcal{X}$ is ***measurable*** if $f^{-1}(\mathcal{O})$ is Lebesgue measurable for every open $\mathcal{O} \subseteq \mathcal{X}$ and is ***locally essentially bounded*** if, for every compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$, there exists a compact set $K \subseteq \mathcal{X}$ such that

$$\lambda\{t \in \mathbb{T} \mid \ f(t) \notin K\} = 0, \tag{2.1}$$

where λ is Lebesgue measure on $\mathbb{R}$. The space of all locally essentially bounded functions defined in $\mathcal{X}$ with domain $\mathbb{T}$ is denoted by $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{X})$.

We will denote by $c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ the set of all sequences $(a_i)_{i\in\mathbb{Z}_{\geq 0}}$ of positive numbers such that $\lim_{i\to\infty} a_i = 0$.

We will make extensive use of recent work on topologies for spaces of sections of vector bundles [Jafarpour and Lewis 2013]. This work, especially as concerns real analytic vector bundles, really makes possible the general and unified setting we provide for handling regularity in geometric control theory. For the purposes of reading what is written in the paper, we simply say that we shall suppose the reader to be familiar with locally convex topologies. We refer to [Conway 1985], for example, as a reference. In particular, we will make use of the fact that the topology of a locally convex space is defined by a family $(p_i)_{i\in I}$ of seminorms. A set B in a topological vector space V is ***von Neumann bounded*** if, for every neighborhood $\mathcal{O} \subseteq V$ of zero, there exists $\lambda \in \mathbb{R}_{>0}$ such that $B \subseteq \lambda\mathcal{O}$.

3. Topologies on spaces of sections

In this section we will define a locally convex topological vector space structure on $\Gamma^\nu(TM)$ for $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$. We separate the discussion into three cases, $\nu \in \mathbb{Z}_{\geq 0}$, $\nu = \infty$, and $\nu = \omega$.

3.1. Fibre norms for jet bundles. An important part is played in our unified treatment of various degrees of regularity by appropriate fibre norms for jet bundles of the tangent bundle. We suppose that the C^r-manifold M has a C^r-Riemannian metric $\mathbb{G}$ and a C^r-affine connection ∇. The existence of these for $r = \infty$ is classical and for $r = \omega$ is proved in [Jafarpour and Lewis 2013, Lemma 2.3]. Let $T^m(T^*M)$ denote the m-fold tensor product of T^*M and let $S^m(T^*M)$ denote the symmetric tensor bundle. First note that ∇ defines a connection on T^*M by duality. Then ∇ defines a connection ∇^m on $T^m(T^*M) \otimes TM$ by asking that the Leibniz Rule be satisfied for the tensor product. Then, for a smooth vector field X, we denote

$$\nabla^{(m)}X = \nabla^m \cdots \nabla^1\nabla X,$$

which is a smooth section of $T^{m+1}(T^*M) \otimes TM$. By convention we take $\nabla^0X = \nabla X$ and $\nabla^{(-1)}X = X$. (The funny numbering makes this agree with the constructions in [Jafarpour and Lewis 2013, §2.1].)

We then have a map

$$\begin{aligned} S^m_\nabla \colon\ J^mTM &\to \oplus_{j=0}^m (S^j(T^*M)\otimes TM)\\ j_mX(x) &\mapsto (X(x), \mathrm{Sym}_1\otimes \mathrm{id}_{TM}(\nabla X)(x), \dots,\\ &\qquad \mathrm{Sym}_m\otimes \mathrm{id}_{TM}(\nabla^{(m-1)}X)(x)), \end{aligned} \tag{3.1}$$

which can be verified to be an isomorphism of vector bundles [Jafarpour and Lewis 2013, Lemma 2.1]. Here $\mathrm{Sym}_m : T^m(V) \to S^m(V)$ is defined by

$$\mathrm{Sym}_m(v_1\otimes\cdots\otimes v_m) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(m)}.$$

Now we note that inner products on the components of a tensor product induce in a natural way an inner product on the tensor product [Jafarpour and Lewis 2013, Lemma 2.2]. Thus, if we suppose that we have a Riemannian metric $\mathbb{G}$ on M, there is induced a natural fibre metric $\mathbb{G}_m$ on $T^m(T^*M)\otimes TM$ for each $m\in\mathbb{Z}_{\ge 0}$. We then define a fibre metric $\overline{\mathbb{G}}_m$ on J^mTM by

$$\begin{aligned} &\overline{\mathbb{G}}_m(j_mX(x), j_mY(x))\\ &\qquad = \sum_{j=0}^m \mathbb{G}_j\Big(\frac{1}{j!}\mathrm{Sym}_j\otimes\mathrm{id}_{TM}(\nabla^{(j-1)}X)(x),\\ &\qquad\qquad \frac{1}{j!}\mathrm{Sym}_j\otimes\mathrm{id}_{TM}(\nabla^{(j-1)}Y)(x)\Big). \end{aligned}$$

(The factorials are required to make things work out with the real analytic topology.) The corresponding fibre norm we denote by $\|\cdot\|_{\overline{\mathbb{G}}_m}$.

3.2. Topology on space of finitely differentiable sections.

In this section we assume that $\nu = m\in\mathbb{Z}_{\ge 0}$. The locally convex topology on the space $\Gamma^m(TM)$ is defined using a family of seminorms.

3.1 Definition: Let $m\in\mathbb{Z}_{\ge 0}$. For a compact set $K\subseteq M$, we define the seminorm p^m_K on $\Gamma^m(TM)$ as

$$p^m_K(X) = \sup\{\|j_mX(x)\|_{\overline{\mathbb{G}}_m} \mid\ x\in K\}.$$

Then the family of seminorms p^m_K, $K\subseteq M$ compact, defines a locally convex topology on $\Gamma^m(TM)$. We call this topology the ***C^m-topology***.

The locally convex topological vector space $\Gamma^m(TM)$ is complete, Hausdorff, separable, and metrizable [Jafarpour and Lewis 2013, §3.4].

3.3. Topology on space of smooth sections.

In the smooth case, i.e., when $\nu=\infty$, the topology we describe has been completely developed in the literature. It is described, for example in [Kriegl and Michor 1997, §41.13]. In [Agrachev and Sachkov 2004] the space of smooth vector fields is topologized by thinking of smooth vector fields as derivations on the algebra $C^\infty(M)$ of smooth functions. The smooth compact-open topology of $C^\infty(M)$, i.e., the topology of uniform convergence of all derivatives on compacts, is used to induce a sort of "weak" topology on the space of smooth vector fields [Agrachev and Sachkov 2004, §2.2]. One can show that this weak topology is the same as C^∞-topology on $\Gamma^\infty(TM)$ that we define now [Jafarpour and Lewis 2013, Theorem 3.5].

3.2 Definition: Suppose that M is a smooth manifold. For $m \in \mathbb{Z}_{\geq 0}$ and for a compact set $K \subseteq M$, we define the seminorm $p^{\infty}_{K,m}$ on $\Gamma^{\infty}(TM)$ by

$$p^{\infty}_{K,m}(X) = \sup\{\|j_m X(x)\|_{\overline{\mathbb{G}}_m} \mid \ x \in K\}.$$

Then the family of seminorms $p^{\infty}_{K,m}$, $K \subseteq M$ compact, $m \in \mathbb{Z}_{\geq 0}$, defines a locally convex topology on $\Gamma^{\infty}(TM)$. We call this topology the ***C^{∞}-topology***.

It can be shown that $\Gamma^{\infty}(TM)$ with C^{∞}-topology is a Hausdorff, separable, and completely metrizable space [Jafarpour and Lewis 2013, §3.2].

3.4. Topology on space of real analytic sections. Now we define a topology on the space of real analytic vector fields. The first observation we make is that $\Gamma^{\omega}(TM)$ is not a closed subspace of $\Gamma^{\infty}(TM)$ with the C^{∞}-topology. Thus the C^{∞}-topology for $\Gamma^{\omega}(TM)$ is not complete [Jafarpour and Lewis 2013, §5], and so it not a suitable topology for analysis.

In this section we describe a suitable topology for $\Gamma^{\omega}(TM)$. There is a bit of history to this topology. For the space of real analytic functions, there are two quite natural topologies. One arises as a direct limit of the compact-open topologies for the holomorphic functions defined on a neighborhood of a complexification of M. Another arises as an inverse limit of holomorphic extensions—again defined in some complexification of M—on a neighborhood of compact subsets of M. It is a hard theorem that these are two descriptions of the same topology [Martineau 1966]. In [Vogt 2013] seminorms for this topology are given in the case of C^{ω}-functions on $\mathbb{R}^n$.

The description of the topology we give is a nontrivial extension of the seminorm construction of [Vogt 2013] to sections of a real analytic vector bundle [Jafarpour and Lewis 2013, §5.2.4].

3.3 Definition: Suppose that M is a real analytic manifold. For every $\mathbf{a} \in c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ and every compact set $K \subseteq M$, we define the seminorm $p^{\omega}_{K,\mathbf{a}}$ on $\Gamma^{\omega}(TM)$ by

$$p^{\omega}_{K,\mathbf{a}}(X) = \sup\{a_0 a_1 \dots a_m \|j_m X(x)\|_{\overline{\mathbb{G}}_m} \mid \ x \in K, \ m \in \mathbb{Z}_{\geq 0}\}.$$

Then the family of seminorms $p^{\omega}_{K,\mathbf{a}}$, $K \subseteq M$ compact, $\mathbf{a} \in c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, define a locally convex vector space structure on $\Gamma^{\omega}(TM)$. We denote this topology by ***C^{ω}-topology***.

It can be shown that $\Gamma^{\omega}(TM)$ endowed with C^{ω}-topology is complete, Hausdorff, separable, and nonmetrizable [Jafarpour and Lewis 2013, §5.3].

3.5. Notation. Now that we have defined locally convex topologies on the vector space $\Gamma^{\nu}(TM)$ for $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, to make the notation simpler we make the following convention. For $K \subseteq M$ be compact, for $k \in \mathbb{Z}_{\geq 0}$, and for $\mathbf{a} \in c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, denote

$$p_K = \begin{cases} p^{\infty}_{K,k}, & \nu = \infty, \\ p^{m}_{K}, & \nu = m, \\ p^{\omega}_{K,\mathbf{a}}, & \nu = \omega. \end{cases} \tag{3.2}$$

The convenience and brevity more than make up for the slight loss of preciseness by using this abbreviated notation.

4. Time-varying vector fields

As we have seen, time-varying vector fields arise naturally by substituting an appropriate control into the equation defining the trajectories for a system. In this section, we first define a special class of time-varying vector fields that we call Carathéodory vector fields of class C^ν. Then using the C^ν-topology on $\Gamma^\nu(TM)$ introduced in Definitions 3.1, 3.2, and 3.3, we give a regularity condition for Carathéodory vector fields of class C^ν so that their flows are C^ν with respect to the initial condition.

4.1 Definition: Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. Suppose that M is a C^r-manifold and $\mathbb{T} \subseteq \mathbb{R}$ is an interval. A map $X : \mathbb{T} \times M \to TM$ is a ***Carathéodory vector field of class C^ν*** if

(i) for every $t \in \mathbb{T}$, the map $X^t : M \to TM$ defined by $X^t(x) = X(t, x)$ is a C^ν-vector field and

(ii) for every $x \in M$, the map $X_x : \mathbb{T} \to TM$ defined by $X_x(t) = X(t, x)$ is Lebesgue measurable.

The set of all Carathéodory vector fields of class C^ν for the time interval $\mathbb{T}$ is denoted by $\mathrm{CF}\Gamma^\nu(\mathbb{T}; TM)$.

4.2 Definition: A Carathéodory vector field $X : \mathbb{T} \times M \to TM$ of class C^ν is ***locally essentially bounded*** if, for every every compact set $K \subseteq M$ and every corresponding seminorm p_K of the form (3.2), there exists $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{>0})$ such that

$$p_K(X^t) \leq g(t), \qquad t \in \mathbb{T}.$$

The set of all locally essentially bounded Carathéodory vector fields of class C^ν is denoted by $\mathrm{LB}\Gamma^\nu(TM)$.

It is obvious that local essential boundedness for Carathéodory vector fields of class C^ν is a joint regularity condition on time and state. Using the C^ν-topology defined on $\Gamma^\nu(TM)$, one can characterize locally essentially bounded Carathéodory vector fields of class C^ν as curves in $\Gamma^\nu(TM)$.

Before stating the theorem characterizing this, suppose that $X : \mathbb{T} \times M \to TM$ is a map such that, for every $t \in \mathbb{T}$, the map $X^t : M \to TM$ defined by $X^t(x) = X(t, x)$ is a vector field of class C^ν. Then we can define a map $\widehat{X} : \mathbb{T} \to \Gamma^\nu(TM)$ by $\widehat{X}(t)(x) = X(t, x)$. Using this definition, one can prove the following theorem.

4.3 Theorem: *Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. Suppose that M is a C^r-manifold and $\mathbb{T} \subseteq \mathbb{R}$ is an interval. Suppose that $X : \mathbb{T} \times M \to TM$ is a map such that, for every $t \in \mathbb{T}$, the map $X^t : M \to TM$ defined by $X^t(x) = X(t, x)$ is a vector field of class C^ν. Then*

(i) $X \in \mathrm{CF}\Gamma^\nu(\mathbb{T}; TM)$ if and only if $\widehat{X} : \mathbb{T} \to \Gamma^\nu(TM)$ is Lebesgue measurable, and

(ii) $X \in \mathrm{LB}\Gamma^\nu(\mathbb{T}; TM)$ if and only if $\widehat{X} : \mathbb{T} \to \Gamma^\nu(TM)$ is locally essentially von Neumann bounded.

Proof: This is the content of Theorems 6.3, 6.9, and 6.21 in [Jafarpour and Lewis 2013]. In all cases, a very good understanding of the locally convex topology plays an important part. ■

Now we want to study the flows of Carathéodory vector fields of class C^ν. As mentioned in the introduction, without imposing any additional condition on these time-varying vector fields, even the local existence of flows and uniqueness of trajectories is not guaranteed. So it is natural to impose some condition on Carathéodory vector fields of class C^ν to ensure that their flows have specific properties. By assuming that the Carathéodory vector field of class C^ν is locally essentially bound, one has the following fundamental result that is proved in [Jafarpour and Lewis 2013] as Theorems 6.6, 6.11, and 6.26. This result in the real analytic case is new, and requires a deep understanding of the C^ω-topology.

4.4 Theorem: *Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $X \in \mathrm{LB}\Gamma^{\nu+1}(\mathbb{T}; TM)$. Then there exist a nonempty set $D_X \subseteq \mathbb{T} \times \mathbb{T} \times M$ and a map $\Phi^X : D_X \to M$ such that*

(i) for every $(t_0, x_0) \in \mathbb{T} \times M$, the set

$$\mathbb{T}_{t_0,x_0} = \{t \in \mathbb{T} \mid \ (t, t_0, x_0) \in D_X\} \tag{4.1}$$

is an open interval;

(ii) the curve $\gamma_{t_0,x_0} : \mathbb{T}_{t_0,x_0} \to M$ defined by

$$\gamma_{t_0,x_0}(t) = \Phi^X(t, t_0, x_0) \tag{4.2}$$

is the unique maximal absolutely continuous solution of the initial value problem

$$\begin{aligned} \gamma'(t) &= X(t, \gamma(t)), \qquad a.e.\ t \in \mathbb{T}_{t_0,x_0}, \\ \gamma(t_0) &= x_0; \end{aligned}$$

(iii) for every $x_0 \in M$ and for every $t, t_0 \in \mathbb{T}$ such that $(t, t_0, x_0) \in D_X$, there exists a neighborhood $\mathcal{U} \subseteq M$ around x_0 such that the map $\eta_{t,t_0} : \mathcal{U} \to M$

$$\eta_{t,t_0}(x) = \Phi^X(t, t_0, x) \tag{4.3}$$

is defined and of class C^ν.

The first two parts of Theorem 4.4 show that the integral curve of locally essentially bounded Carathéodory vector fields of class $C^{\nu+1}$ passing through a fixed point, exists locally and is unique. The last part shows that the flows of a locally essentially bounded Carathéodory vector field of class $C^{\nu+1}$ is of class C^ν in initial condition. In particular, flows of vector fields in $\mathrm{LB}\Gamma^\nu(\mathbb{T}; TM)$ for $r \in \{\infty, \omega\}$ depend on initial condition in a C^ν manner. We comment that the theorems in [Jafarpour and Lewis 2013] are stated for a slightly larger class of vector fields than locally essentially bounded, namely those that are termed "locally integrally bounded." Also, in [Jafarpour and Lewis 2013] regularity of class "$\nu + 1$" is replaced with the slightly weaker regularity of class "$\nu + \mathrm{lip}$."

5. Control systems

In this section we define a class of control systems that are called C^ν-control systems. We consider a control system as a parametrized family of vector fields, parametrized by elements of control set. Therefore, in order to define this class of control systems, we need a notion of parametrized vector fields of class C^ν, where the parameter take its value in a control set.

5.1 Definition: Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. Suppose that M is a C^r-manifold and $\mathcal{P}$ is a topological space.

(i) A map $F : \mathcal{P} \times M \to TM$ is a ***separately parametrized vector field of class C^ν*** if
  (a) for every $p \in \mathcal{P}$, the map $F^p : M \to TM$ defined by $F^p(x) = F(p, x)$ is a C^ν-vector field and
  (b) for every $x \in M$, the map $F_x : \mathcal{P} \to TM$ defined by $F_x(p) = F(p, x)$ is continuous.

The set of all separately parametrized vector fields of class C^ν is denoted by $\mathrm{SP}\Gamma^\nu(\mathcal{P}; TM)$.

(ii) A map $F : \mathcal{P} \times M \to TM$ is ***jointly parametrized vector field of class C^ν*** if
  (a) it is separately parametrized of class C^ν and
  (b) for every $(p_0, x_0) \in \mathcal{P} \times M$ and for every $\epsilon > 0$, there exist a relatively compact neighborhood $\mathcal{U} \subseteq M$ of x_0, a seminorm $p_{\mathrm{cl}(\mathcal{U})}$ of the form (3.2), and a neighborhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that

$$p_{\mathrm{cl}(U)}(F^p - F^{p_0}) < \epsilon, \qquad p \in \mathcal{O}.$$

The set of all jointly parametrized vector fields of class C^ν with the topological vector space $\mathcal{P}$ is denoted by $\mathrm{JP}\Gamma^\nu(\mathcal{P}, TM)$.

A jointly parametrized vector field of class C^ν has the nice property that, upon replacing the parameter by a locally essentially bounded control, one gets a locally essentially bounded Carathéodory vector field of class C^ν. Before proving this result, we state an obvious characterization of jointly parametrized vector fields of class C^ν.

5.2 Theorem: *Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. Suppose that M is a C^r-manifold and $\mathcal{P}$ is a topological space. Suppose that the map $F : \mathcal{P} \times M \to TM$ is such that*

$$F^p \in \Gamma^\nu(TM), \qquad p \in \mathcal{P}.$$

We define the map $\widehat{F} : \mathcal{P} \to \Gamma^\nu(TM)$ as $\widehat{F}(p) = F^p$. Then $F \in \mathrm{JP}\Gamma^\nu(\mathcal{P}, TM)$ if and only if $\widehat{F}$ is a continuous map, considering the C^ν-topology on $\Gamma^\nu(TM)$.

We comment that in [Jafarpour and Lewis 2013, §7.1] more concrete descriptions of jointly parameterized vector fields of class C^ν are given.

We now have the following useful result.

5.3 Theorem: *Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. Suppose that M is a C^r-manifold and $\mathcal{P}$ is a topological space. Suppose that $F : \mathcal{P} \times M \to TM$ is a jointly parametrized vector field of class C^ν and $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{P})$. Then the map $F^\mu : \mathbb{T} \times M \to TM$ defined by*

$$F^\mu(t, x) = F(\mu(t), x)$$

has the property that $F^\mu \in \mathrm{LB}\Gamma^\nu(\mathbb{T}; TM)$.

Proof: We prove this theorem for the case $\nu = \omega$. The proof for the case $\nu = \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is similar.

By Theorem 4.3, it suffices to show that $\widehat{F^\mu} : \mathbb{T} \to \Gamma^\omega(TM)$ is a measurable and locally essentially bounded map. Note that we have

$$\widehat{F}^\mu(t)(x) = \widehat{F} \circ \mu(t)(x), \qquad x \in M,\ t \in \mathbb{T}.$$

So we have $\widehat{F}^{\mu} = \hat{F} \circ \mu$.

By Theorem 5.2, since $F : \mathcal{P} \times M \to TM$ is a jointly parametrized vector field of class C^{ω}, the map $\widehat{F} : \mathcal{P} \to \Gamma^{\omega}(TM)$ is continuous. Since μ is measurable and $\widehat{F}$ is continuous, the map $\hat{F} \circ \mu = \hat{F}^{\mu}$ is measurable. Also, since $\mu \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{T}; \mathcal{P})$, for every compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$ there exists a compact set $K \subseteq \mathcal{P}$ such that

$$\lambda\{t \in \mathbb{T}' \mid \ \mu(t) \notin K\} = 0.$$

Since $\widehat{F}$ is continuous, $\widehat{F}(K)$ is compact. We denote $K' = \widehat{F}(K)$. Note that, if $t \in \mathbb{T}'$ is such that $\mu(t) \in K$, then $\widehat{F} \circ \mu(t) \in K'$. Since we have $\widehat{F} \circ \mu = \widehat{F}^{\mu}$, we can write

$$\{t \in \mathbb{T}' \mid \ \widehat{F}^{\mu} \notin K'\} \subseteq \{t \in \mathbb{T}' \mid \ \mu(t) \notin K\}.$$

This means that

$$\lambda\{t \in \mathbb{T}' \mid \ \widehat{F}^{\mu} \notin K'\} \leq \lambda\{t \in \mathbb{T}' \mid \ \mu(t) \notin K\} = 0.$$

So we have

$$\lambda\{t \in \mathbb{T}' \mid \ \widehat{F}^{\mu}(t) \notin K'\} = 0.$$

This shows that $\widehat{F}^{\mu}$ is locally essentially bounded. ■

Using the notion of jointly parametrized vector fields of class C^{ν} in Definition 5.1, one can define a C^{ν}-control system.

5.4 Definition: Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. A ***C^{ν}-control system*** is a triple $\Sigma = (M, \mathcal{C}, F)$ such that
- (i) M is a C^{r}-manifold called ***state manifold***,
- (ii) $\mathcal{C}$ is a topological space called ***control space***, and
- (iii) $F : \mathcal{C} \times M \to TM$ is a jointly parametrized vector field of class C^{ν}.

When a new class of control systems is introduced, it is interesting to see how large this class is, comparing to the other classes of control systems. One may expect that the class of C^{ν}-control systems contains most of the interesting real analytic control systems that we know. The following example shows that control-affine systems are C^{ν}-control systems.

5.5 Example: Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. Suppose that M is a C^{r}-manifold, $f_0, f_1, f_2, \dots, f_k \in \Gamma^{\nu}(TM)$, and $\mathcal{C} \subseteq \mathbb{R}^{k}$. We define the map $F : \mathcal{C} \times M \to TM$ by

$$F(\mathbf{u}, x) = f_0(x) + \sum_{i=1}^{n} u_i f_i(x).$$

To show that F is a jointly parametrized vector field of class C^{ν}, according to Theorem 5.2 we must show that the map $\widehat{F}$ is continuous. This, however, follows easily. We first consider the map

$$\begin{aligned} \Lambda_F \colon \ \mathbb{R}^{k+1} &\to \Gamma^{\nu}(TM) \\ (u_0, u_1, \dots, u_k) &\mapsto \sum_{i=0}^{n} u_i f_i(x). \end{aligned}$$

This being a linear map from a finite-dimensional vector space, it is continuous [Horváth 1966, Proposition 2.10.2]. It follows immediately that $\widehat{F}$ is continuous.

Using Theorems 4.4 and 5.3, one has the following result for trajectories of C^ν-control systems. We draw attention to the fact that this is the first such result that has been obtained for real analytic systems.

5.6 Corollary: *Let $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $r \in \{\infty, \omega\}$, as required. Suppose that $\Sigma = (M, \mathcal{C}, F)$ is a $C^{\nu+1}$-control system and $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ is a control. Then, for each $(t_0, x_0) \in \mathbb{T} \times M$, the open-loop vector field F^μ possesses a unique maximal absolutely integral curve γ satisfying $\gamma(t_0) = x_0$. Moreover, the resulting flow is C^ν in initial condition.*

6. Discussion

What we have provided in the preceding development is a unified framework for handling regularity in geometric control theory, significantly including real analytic systems in this framework. In order to provide some context for what we have done, in this section we overview the manner in which regularity has been treated in the literature. As we shall see, the treatment of this, especially in the geometric control literature, is a little helter-skelter, and we hope that the framework we give here can provide a little organization to how this is typically done.

First of all, we mention that the treatment of regularity in the literature on stabilization is typically presented in a coherent manner, with minimal regularity hypotheses. (This minimality is both a strength and a limitation.) In this area, the state space is typically $\mathbb{R}^n$ and the analysis framework is closely connected with the theory of ordinary differential equations. Thus hypotheses here are closely connected with standard results in the theory of differential equations. For example, in [Sontag 1998, Page 43] systems are considered to be (in our terminology) $\mathrm{JP}\Gamma^1$. A common assumption in this literature is for a system to be $\mathrm{JP}\Gamma^{\mathrm{lip}}$, in keeping with the standard Lipschitz condition for uniqueness of solutions for differential equations, e.g., [Clarke, Ledyaev, Sontag, and Subotin 1997]. (We have not considered Lipschitz regularity here, but this can be done just as we have done in the finitely differentiable, smooth, and real analytic cases [Jafarpour and Lewis 2013, §3.5].) We will mention here that our constructions do not make any assumptions on the nature of the topology of the control set; often this is assumed to be a metric space, or to have other restrictive properties, even being assumed to be a subset of Euclidean space, e.g., [Bressan and Piccoli 2007], which is a somewhat unnatural condition mathematically for systems that are not control-affine.

In contrast to the stabilization literature, in the geometric control literature, the treatment of regularity is not at all unified. This is perhaps because of the absence of a framework (such as we provide here) for treating regularity in a consistent manner. Let us point out a few of the sorts of conditions we encountered on a quick scan of the literature. We should say, however, that there are about as many ways of stating regularity hypotheses as there are papers where these hypotheses are stated clearly.

It is not uncommon for the only assumption to be made is that the parameterized vector field $F : \mathcal{U} \times M \to TM$ is regular (say, smooth) when the control value is fixed [Jurdjevic 1997, Page 21]. Such an hypothesis is workable for control-affine systems (cf. Example 5.5) or if one is dealing only with piecewise constant controls. However, if one works with measurable controls, it is necessary to specify appropriate joint hypotheses on state and control.

One way in which regularity is handled is to assume that the control set is an open subset of $\mathbb{R}^m$ and that F is regular in the normal sense, since now one can differentiate with respect to control. This is done, for example, in [Lee and Markus 1967, Page 31] and [Bonnard and Chyba 2003, Page 37] in the C^1 case, in [Coron 1994] in the smooth setting, and in [Sontag 1992] in the real analytic setting. The assumption that the control set is such that it permits differentiation with respect to control is problematic, even when the control take values in Euclidean space, since control sets typically have boundaries that make differentiation unclear. Of course, this can be handled by an *ad hoc* extension to a neighborhood of the control set.

Mixing of regularity conditions is also common. In [Sussmann 1979] it is pointed out that, for piecewise constant controls, joint dependence conditions are not required, but for measurable controls such conditions are required. For real analytic systems, the condition $\mathrm{SP}\Gamma^{\omega} \cap \mathrm{JP}\Gamma^{\mathrm{lip}}$ is suggested as at least giving existence and uniqueness of trajectories for open-loop controls (it will not give C^{ω} dependence on initial conditions, however). In [Isaiah 2011] the hypotheses are, in our language, $\mathrm{SP}\Gamma^{\omega} \cap \mathrm{JP}\Gamma^{1}$. In [Grasse 1992] systems are considered in $\mathrm{SP}\Gamma^{r} \cap \mathrm{JP}\Gamma^{1}$ for $r \in \{\infty, \omega\}$. In [Sussmann 1998] this mixing of regularity is formulated clearly, and systems are considered that are, in our language, in $\mathrm{SP}\Gamma^{k} \cap \mathrm{JP}\Gamma^{l}$ for $k, l \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with $k \geq l$. (The equivalence of the conditions given in [Sussmann 1998] and the conditions stated in our notation is proved as Propositions 7.2 and 7.5 in [Jafarpour and Lewis 2013].) This mixing is also present in [Agrachev and Gamkrelidze 1993] where real analytic vector fields are treated as derivations on the algebra of *smooth* functions, although [Grabowski 1981] proves the equivalence of real analytic vector fields with derivations on the algebra of *real analytic* functions. In [Jafarpour and Lewis 2013, §5.4] the connection between real analytic vector fields and derivations of real analytic functions is used to provide a “weak” characterization of the C^{ω}-topology, like that for the C^{∞}-topology in [Agrachev and Sachkov 2004, §2.2].

In general, the treatment of real analyticity in the control literature has been done in an *ad hoc* manner. Real analytic time-varying vector fields are studied in [Agrachev and Gamkrelidze 1978], and topological considerations are given similar in spirit to what we do here. However, the analysis is carried out only in Euclidean space, and for real analytic vector fields admitting an holomorphic extension to a neighborhood of fixed width in $\mathbb{R}^n$. By contrast, our treatment of real analytic time-varying vector fields in Section 4 is global and general. As mentioned above, the treatment of the real analytic case in [Agrachev and Gamkrelidze 1993] is done by working with derivations of smooth functions. In our framework, we can carry this out within the real analytic setting [Jafarpour and Lewis 2013, §5.4]. In [Sussmann 1998] a rather stringent definition is provided for a real analytic control system, involving subanalytic sets.

In contrast to the preceding survey of the literature concerning regularity of control systems, the approach we provide in this paper is comprehensive and unified across varying classes of regularity.

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