Control systems and locally convex topologies\textsuperscript{1}

Saber Jafarpour\textsuperscript{2}

Queen’s University

Meeting on System and Control Theory,
Waterloo, 5-6 May 2014

\textsuperscript{1}Joint work with Professor Andrew D. Lewis
\textsuperscript{2}PhD student in Department of Mathematics and Statistics, Queen’s University, Kingston, ON, Canada
In geometric control theory, a control system is described by the following differential equation

$$\dot{x} = f(u, x),$$

where the right hand side is a parametrized family of vector fields $f : \mathcal{U} \times M \to TM$, with $\mathcal{U}$ being the control set.

The trajectories of the control system are the solutions of this differential equation for a locally essentially bounded control $u(\cdot)$. 
In the literature, there are many different regularity assumptions on $f$.

In one approach,\(^3\) it is assumed that the control set $\mathcal{U}$ is a topological space and the parametrized vector field $f : \mathcal{U} \times M \to TM$ has first derivatives continuous with respect to $x$ and $u$.

Although this is a general and coherent approach, but it has the deficiency of not accounting for stronger regularity when it is present.

\(^3\)For example in the book “Mathematical Control Theory” by Sontag.
In another approach,\textsuperscript{4} it is assumed that the control set $\mathcal{U}$ is an open subset of Euclidean space and the parametrized vector field $f : \mathcal{U} \times M \rightarrow TM$ is of class $C^\nu$, for $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \cup \{\omega\}$.

This approach includes general regularity classes, but it is restrictive in terms of control sets (the control set is an open subset of $\mathbb{R}^k$).

It seems that there is no coherent approach for studying different regularity classes of control systems in the literature.

\textsuperscript{4}For example in the book “Foundation of Optimal Control Theory” by E. B. Lee & L. Markus
In this talk we give a unified framework for studying regularity class $C^{\nu}$, for $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \cup \{\omega\}$.

In particular, our framework includes the real analytic class.

In order to construct such a framework, we first assume that

1. The control set $\mathcal{U}$ is an arbitrary topological space.
2. The parametrized vector field $f : \mathcal{U} \times M \to TM$ is of class $C^{\nu}$ with respect to $x$, when $u$ is fixed.

We call this a $C^{\nu}$-parametrized vector field.
Idea: Consider $C^\nu$-parametrized vector fields $f$ as maps from the space of parameters to the space of vector fields.

We denote by $\Gamma^\nu(TM)$ the set of all vector fields of class $C^\nu$ on $M$.

$\Gamma^\nu(TM)$ is a vector space.

Correspondence

If $f : \mathcal{U} \times M \to TM$ is a $C^\nu$-parametrized vector field, then the corresponding map $\hat{f} : \mathcal{U} \to \Gamma^\nu(TM)$ is defined as

$$\hat{f}(u)(x) = f(u, x).$$

In order to impose useful conditions on $\hat{f}$, we will use a topology on $\Gamma^\nu(TM)$. 
Locally convex topologies

Locally convex space

A **locally convex space**, is a vector space \( V \) equipped with a family of seminorms \( \{ p_{\alpha} \}_{\alpha \in A} \).

- **Comparison**: Locally convex spaces can be considered as a generalization of normed spaces.
- Similar to normed spaces, one can define a topology on a locally convex space using seminorms.
- One can define similar notions such as boundedness, continuity and measurability for locally convex spaces.
Suppose that $V$ is a vector space with norm $\| \cdot \|$ and $U$ is a topological space.

**Continuity**

A map $f : U \to V$ is **continuous at** $u \in U$ if for every $\epsilon > 0$, there exists a neighbourhood $N_u$ of $u$ such that

$$\| f(v) - f(u) \| < \epsilon, \quad \forall v \in N_u$$
Suppose that $V$ is a locally convex space with seminorms $\{p_\alpha\}_{\alpha \in A}$ and $U$ is a topological space.

**Continuity**

A map $f : U \rightarrow V$ is **continuous at** $u \in U$ if for every $\alpha \in A$ and every $\epsilon > 0$, there exists a neighbourhood $N_u$ of $u$ such that

$$p_\alpha(f(v) - f(u)) < \epsilon, \quad \forall v \in N_u$$
We define a locally convex structure on $\Gamma^\nu(TM)$ using a family of seminorms.

- For defining the locally convex structure on $\Gamma^\nu(TM)$, we separate the cases $\nu \in \mathbb{Z}_{>0}$, $\nu = \infty$ and $\nu = \omega$.
- If $\xi \in \Gamma^\nu(TM)$, then $j_m\xi(x)$ can be considered as the first $m$ terms in Taylor series of $\xi$ around $x$.
- We define a fiber norm $\| \cdot \|$ on the space of jets in a specific way (not presented here).
- We define $\mathbf{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ as

$$\mathbf{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0}) = \{(a_0, a_1, a_2, \ldots) \mid a_i \in \mathbb{R}_{\geq 0}, \lim_{i \to \infty} a_i = 0\}.$$
Locally convex topologies on space of vector fields

The $\text{CO}^\nu$-structure on $\Gamma^\nu(TM)$ is the locally convex structure on $\Gamma^\nu(TM)$ defined using the seminorms,

**Cases $\nu \in \mathbb{Z}_{>0}$**

$$p^K_\nu(\xi) = \sup\{\|j_\nu \xi(x)\| \mid x \in K\}, \ K \subseteq M \text{ compact};$$

**Case $\nu = \infty$**

$$p^K_\infty(m)(\xi) = \sup\{\|j_m \xi(x)\| \mid x \in K\}, \ m \in \mathbb{Z}_{\geq 0}, \ K \subseteq M \text{ compact};$$

**Case $\nu = \omega$**

$$p^K_\omega(a)(\xi) = \sup\{a_0a_1 \cdots a_m\|j_m \xi(x)\| \mid x \in K, m \in \mathbb{Z}_{\geq 0}\},
\quad a = (a_0, a_1, \ldots) \in c_0(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0}), \ K \subseteq M \text{ compact}.$$
$C^\nu$-control systems

- Using the $CO^\nu$-topology, we can define a $C^\nu$-control system as

**$C^\nu$-control system**

A $C^\nu$-control system is a triple $\Sigma = (M, f, U)$, where

1. $M$ is a differentiable manifold,
2. $U$ is a topological space, and
3. $f : U \times M \to TM$ is a $C^\nu$-parametrized vector field such that $\hat{f} : U \to \Gamma^\nu(TM)$ is continuous in $CO^\nu$-topology.

- The third condition is a checkable condition, using the seminorms for $CO^\nu$-topology on $\Gamma^\nu(TM)$.
Main Theorem

These $C^{\nu}$-topologies helps us to prove the following fundamental result.

Theorem

Consider the control system

$$\dot{x} = f(u, x),$$

where $f : \mathcal{U} \times M \to TM$ is a $C^{\nu}$-parametrized vector field for $\nu \in \mathbb{Z}_{>0} \cup \{\infty\} \cup \{\omega\}$. If the curve $\hat{f} : \mathcal{U} \to \Gamma^{\nu}(TM)$ is continuous in $C^{\nu}$-topology on $\Gamma^{\nu}(TM)$, then the trajectory of the system starting at $x_0$ exists, is unique and is $C^{\nu-1}$ dependent on the $x_0$.

This result relies on a deep and difficult theorem about time-varying vector fields$^5$.

A classical result

One can show that for $\nu = 1$, our main theorem is just the classical existence and uniqueness result for $M = \mathbb{R}^n$.

Existence and Uniqueness Theorem

Suppose that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$-parametrized vector field on $\mathbb{R}^n$, $\mu : \mathbb{T} \to \mathbb{R}$ is a locally essentially bounded curve and $M, N_1, N_2, \ldots, N_n > 0$ such that

\[
|f(\mu(t), x)| \leq M,
\]

\[
\left| \frac{\partial f}{\partial x^j}(\mu(t), x) \right| \leq N_j, \quad \forall j \in \{1, 2, \ldots, n\},
\]

holds for almost every $t$, in a neighbourhood of $x_0$. Then the trajectory of the system for the control $\mu$ starting at $x_0$ exists, is unique and depends continuously on the initial condition.
Control-affine systems with vector fields in $\Gamma^\nu(TM)$ are $C^\nu$-control systems.

Consider a control-affine system with $f : \mathbb{R}^m \times M \rightarrow TM$ defined as

$$f(u, x) = f_0(x) + \sum_{i=1}^{m} u^i f_i(x),$$

such that $f_i \in \Gamma^\nu(TM)$ for every $i \in \{0, 1, \ldots, m\}$. One can show that $\hat{f} : \mathbb{R}^m \rightarrow \Gamma^\nu(TM)$ is continuous in $CO^\nu$-topology, so $\Sigma = (M, f, \mathbb{R}^m)$ is a $C^\nu$-control system. So trajectories for control-affine systems depend in a regular manner on initial conditions when an open-loop control has been fixed.